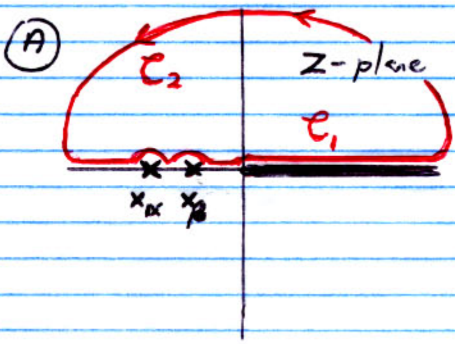
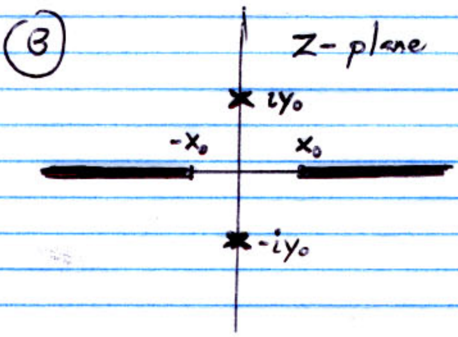
REMARKS ON CONTOUR INTEGRATION for DISPERSION RELATIONS & GREEN FNS.

The techniques used to determine the structure of Green functions, T-matrices, etc., in the complex plane are just a straightforward application of Cauchy's theorem, but there are some manoeuvres that it is useful to spell out explicitly.

As discussed in the notes, we are interested in the general behaviour of a complex variable whose singularities consist of branch cuts on some portions of the real axis, along with discrete poles. In the notes these poles were also on the real axis, and so we are interested in functions with the pole structure shown in Fig (A) at left. However we will find it interesting to also look at the class of functions with the structure shown in Fig (B), where there are also poles off the real axis.





We wish to find the behaviour throughout the complex plane of functions $f(z)$ with these pole structures. Of course in all cases we start from Cauchy's theorem, viz., that

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz' \, \frac{f(z')}{z'-z} \tag{1}$$

It is worth stopping a moment to dwell on how extraordinary is this formula. Those who first meet it are often amazed that the value of $f(z)$ could be determined everywhere inside the contour $\mathcal{C}$, no matter what the contour or the value of z, provided that $f(z)$ is assumed regular inside $\mathcal{C}$ (ie., has no singularities inside $\mathcal{C}$). The same people are often quite unsurprised to hear that in simple electrostatics, the electric potential inside a closed boundary is entirely determined by the potential on the boundary – or that the fluid flow of an incompressible inviscid ideal fluid is determined by the flow at a surface around it (provided there are no charges or vortices inside these surfaces). And yet these results are all equivalent. It is hard to know which is more remarkable – the formula (1), or the fact that some physical systems are described so accurately by it.

Now consider the contour shown in Fig. (A); we divide it into 2 sections, so that $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$, where $\mathcal{C}_1$ is the portion along the real axis, and $\mathcal{C}_2$ is the portion of the contour off at ∞ in the upper half-plane.

Let's suppose first that $f(z)$ has no singularities at all in the upper half-plane. Then eqtn (1) applies directly to any point in the upper half-plane. Now let us consider the function

$$f^+(x) = \lim_{\delta \to 0} f(x + i\delta) \tag{2}$$

Then quite clearly we have

$$2\pi i f^+(x) = \lim_{\delta \to 0} \oint_{\mathcal{C}} \frac{f(z')}{z' - x - i\delta} dz' \tag{2}$$

$$= \lim_{\delta\to 0} \int dx' \frac{f(x')}{x' - x + i\delta} + \int_{\mathcal{C}_2} dz' \frac{f(z')}{z' - x} = f_1^+ + f_2^+ \tag{3}$$

where in f_2^+ we ignore the $i\delta$ factor since it has no influence for any finite x.
The first integral is just

$$\left.\begin{aligned} 2\pi i f_1^+(x) &= \lim_{\delta \to 0} \int dx' \frac{f^+(x')}{x' - x + i\delta} \\ &= \mathbb{P} \int_{-\infty}^{\infty} dx' \frac{f^+(x')}{x' - x} + i\pi f^+(x) \end{aligned}\right\} \tag{4}$$

where we have used the rule, already explained to you in the supplement, that we can use the identity

$$\frac{1}{x' - x \pm i\delta} \equiv \mathbb{P}\left(\frac{1}{x' - x}\right) \mp i\pi\,\delta(x' - x) \tag{5}$$

as an operator identity in dealing with integrals of this type (cf eqtns A-17 and A-18 of the supplement).

Now consider the integral along $\mathcal{C}_2$. Let us write $z' = r' e^{i\theta'}$ on this large circle, and assume that $r' \to \infty$. Then we have

$$2\pi i f_2^+ = \int_{\mathcal{C}_2} dz' \frac{f(z')}{z' - x} \leq \mathrm{Max}(f(z')) \int d\theta' \frac{i r' e^{i\theta'}}{r' e^{i\theta'}} = i\pi\, \mathrm{Max}(f(z'))\Big|_{z' \in \mathcal{C}_2} \tag{6}$$

However we are in all cases interested in problems where

$$\lim_{|z| \to \infty} |f(z)| = 0 \tag{7}$$

It then follows that

$$f_2^+(x) = 0 \tag{8}$$

and so

$$f^+(x) = f_1^+(x) = \frac{1}{2\pi i}\left[\mathbb{P}\int_{-\infty}^{\infty} dx' \frac{f^+(x')}{x' - x} + i\pi f^+(x)\right] \tag{9}$$

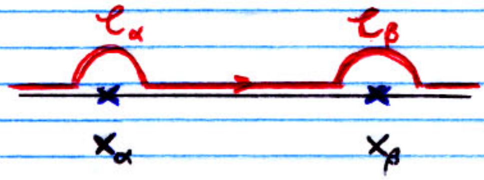
Now let's go back to the full problem shown in Fig. (A). We will assume that $f(z)$ is specified entirely by its singularities, of the following kind:

$$\left.\begin{aligned} &\text{Simple poles at } z = x_\alpha, x_\beta \text{ with residues } A_\alpha, A_\beta. \\ &\text{Branch cut from } x = 0 \text{ to } x \to \infty \text{ with magnitude } A(x) = \mathrm{Im}\, f^+(x) \end{aligned}\right\} \tag{10}$$

We again assume that $f(z)$ vanishes as $|z| \to \infty$, and so concentrate again on the contour $\mathcal{C}_1$. Let's do this very carefully. We have, for z in the upper ½-plane

$$2\pi i f(z) = \fint_{-\infty}^{\infty} dx' \frac{f^+(x')}{x'-z} + \int_{\mathcal{C}_\alpha} dx' \frac{f^+(x')}{x'-z} + \int_{\mathcal{C}_\beta} dx' \frac{f^+(x')}{x'-z} \qquad (11)$$

where the integral $\fint_{-\infty}^{\infty} dx'$ excludes the 2 poles, ie., comes to within a distance ϵ of them on either side. Let's now look at the integrals around these poles — we have the situation shown, where a semicircle of radius ϵ is traced around the poles.



Consider the value of the integral around the pole at x_α. We have

$$\left.\begin{aligned} \int_{\mathcal{C}_\alpha} dx' \frac{f^+(x')}{x'-z} &= \int_{\mathcal{C}_\alpha} dx' \frac{f(x_\alpha + \epsilon e^{i\theta'})}{x_\alpha + \epsilon e^{i\theta'} - z} \\ &= \int_{\pi}^{0} d\theta' \, i\epsilon e^{i\theta'} \frac{A_\alpha}{\epsilon e^{i\theta'}} \frac{1}{x_\alpha - z} \\ &= -i\pi \left(\frac{A_\alpha}{x_\alpha - z} \right) \end{aligned}\right\} \qquad (12)$$

where we notice that the negative sign comes from the fact that we are taking the contour $\mathcal{C}_\alpha$ along the counter-clockwise direction; and we note that the value of $f(x')$ on the contour is just $A_\alpha / \epsilon e^{i\theta'}$.

Thus we now have

$$2\pi i f(z) = \fint dx' \frac{f^+(x')}{x'-z} - i\pi \left[\frac{A_\alpha}{x_\alpha - z} + \frac{A_\beta}{x_\beta - z} \right] \qquad (13)$$

Now let $z \to x + i\delta$; then: $$\fint dx' \frac{f^+(x')}{x'-x-i\delta} = \mathcal{P} \fint_{-\infty}^{\infty} dx' \frac{f^+(x')}{x'-x} + i\pi f^+(x) \qquad (14)$$

and since the LHS of (13) when $z \to x + i\delta$ is just $2\pi i f^+(x)$, we finally get

$$f^+(x) = \frac{A_\alpha}{x - x_\alpha} + \frac{A_\beta}{x - x_\beta} + \frac{1}{i\pi} \mathcal{P} \fint_{-\infty}^{\infty} dx' \frac{f^+(x')}{x'-x} \qquad (15)$$

To get more from this result it is useful to look again at the Fig. (A). We see that as far as the singularities are concerned, there is a reflection symmetry about the real axis, and so it makes sense to assume that for all regions of the real axis except at the poles and branch cuts, we have

$$\mathrm{Im}\, f(x) = 0 \qquad (\text{except poles \& branch cuts}). \qquad (16)$$

Note that in the electrostatic analogy this would be obvious, since all electric field lines on the real axis would be along the axis, where there was no charge (again by symmetry), assuming no charge sources at ∞.

With assumption (16), we can assume that away from the poles and branch cuts, $f^+(x)$ is entirely real. We may then write, from (15), that

$$f^+(x) = \frac{A_\alpha}{x-x_\alpha} + \frac{A_\beta}{x-x_\beta} + \frac{1}{i\pi}\mathbb{P}\int_{-\infty}^{\infty} dx' \frac{\mathrm{Re}\, f^+(x')}{x'-x} + \frac{1}{\pi}\mathbb{P}\int_0^{\infty} dx' \frac{\mathrm{Im}\, f^+(x')}{x'-x} \qquad (17)$$

Let's now take the real and imaginary parts of (17); we get:

$$\left.\begin{aligned} \mathrm{Re}\, f^+(x) &= \frac{A_\alpha}{x-x_\alpha} + \frac{A_\beta}{x-x_\beta} + \frac{1}{\pi}\mathbb{P}\int_0^{\infty} dx' \frac{\mathrm{Im}\, f^+(x')}{x'-x} \\ &= \frac{A_\alpha}{x-x_\alpha} + \frac{A_\beta}{x-x_\beta} + \frac{1}{\pi}\mathbb{P}\int_0^{\infty} dx' \frac{A(x')}{x'-x} \end{aligned}\right\} \qquad (18)$$

and

$$\mathrm{Im}\, f^+(x) = A(x) = -\frac{1}{\pi}\mathbb{P}\int_{-\infty}^{\infty} dx' \frac{\mathrm{Re}\, f^+(x')}{x'-x} \qquad (19)$$

These relationships are known as "dispersion relations" (they were first derived in optics by Kramers & Kronig).

Now let's go back to (9). In the same way as above, we see it gives

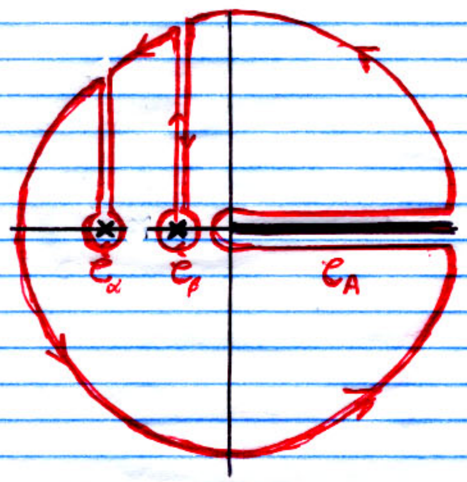
$$\mathrm{Im}\, f(x+i\delta) = f^+(x) = \frac{\mathbb{P}}{i\pi}\int_{-\infty}^{\infty} dx' \frac{f^+(x')}{x'-x} \qquad (20)$$

so that

$$\left.\begin{aligned} \mathrm{Re}\, f^+(x) &= \frac{1}{\pi}\mathbb{P}\int_{-\infty}^{\infty} dx' \frac{\mathrm{Im}\, f^+(x')}{x'-x} \\ \mathrm{Im}\, f^+(x) &= -\frac{1}{\pi}\mathbb{P}\int_{-\infty}^{\infty} dx' \frac{\mathrm{Re}\, f^+(x')}{x'-x} \end{aligned}\right\} \qquad (21)$$

and these relations are sometimes also known as "Kramers-Kronig" relations (by physicists), or "Hilbert transforms" (by mathematicians).

Note that we could have obtained the results (15) - (19) by choosing another contour (we will still have to add (16) as an assumption). Let us now consider the contour in which we circle the whole plane, still avoiding the singularities. We use the contour shown in Fig. (c). This extends in a complete circle at $|z| \to \infty$, and we will again assume that the integral at ∞ gives 0, using the same argument as before. We see also that the contributions coming from the straight lines coming into the poles at x_α and x_β give zero contribution, since the incoming & outgoing sections cancel.



We are thus left we with

$$
\left.\begin{aligned}
2\pi i f(z) &= 2\pi i \left[\frac{A_\alpha}{z-x_\alpha} + \frac{A_\beta}{z-x_\beta} \right] \\
&\quad + \int_0^\infty dx' \left[\frac{f^+(x')}{x'-z+i\delta} - \frac{f^-(x')}{x'-z-i\delta} \right]
\end{aligned}\right\} \qquad (22)
$$

$$\text{where } f^\pm(x) = f(x \pm i\delta) \qquad (23)$$

Now using reflection symmetry, as above, we see that

$$f^+(x) = \left(f^-(x)\right)^* \qquad (30)$$

so that

$$
\left.\begin{aligned}
f^+(x) - f^-(x) &= 2i\, \mathrm{Im}\, f^+(x) \\
&= 2i\, A(x)
\end{aligned}\right\} \qquad (31)
$$

so that

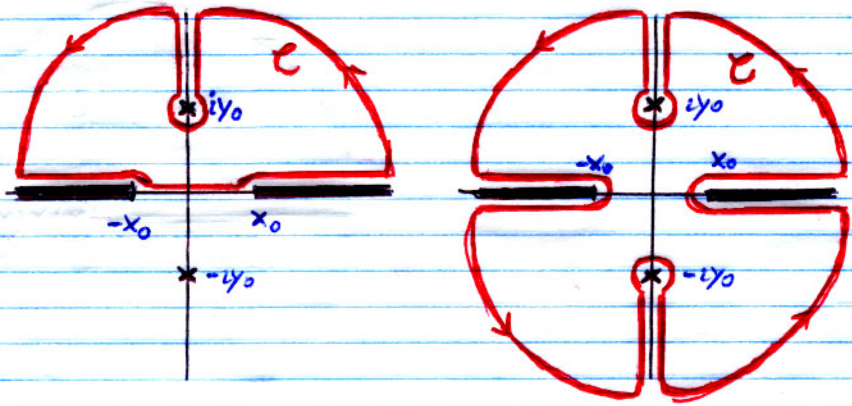
$$
\left.\begin{aligned}
f(z) &= \frac{A_\alpha}{z-x_\alpha} + \frac{A_\beta}{z-x_\beta} + \frac{1}{\pi}\int_0^\infty dx' \frac{\mathrm{Im}\, f^+(x)}{x'-x} \\
&= \frac{A_\alpha}{z-x_\alpha} + \frac{A_\beta}{z-x_\beta} + \frac{1}{\pi}\int_0^\infty dx' \frac{A(x')}{x'-x}
\end{aligned}\right\} \qquad (32)
$$

which is essentially the form used in the notes (eqtn (349) of section B). If we now let $z \to x + i\delta$ we get

$$f^+(x) = \frac{A_\alpha}{x-x_\alpha} + \frac{A_\beta}{x-x_\beta} + \frac{1}{\pi}\left[\mathcal{P}\int_0^\infty dx' \frac{\mathrm{Re}\, f^+(x')}{x'-x} + i\pi\,\Theta(x)\, \mathrm{Im}\, f^+(x) \right] \qquad (33)$$

where we notice that the $\Theta(x)$ function in (33) is actually unnecessary because of (16). If we now take the real part of (33), we get (18), and the imaginary part gives (19).

Let us finally add a few words on the problem depicted in Fig (3) on page 1. It will be clear from above that we have various choices for the contour to be used in dealing with this system. Each of these contours has certain advantages. Physically contours like this can arise when one considers not only the usual real states in the complex energy plane, but also introduces in a phenomenological way a pole on the imaginary axis to represent an instability of the system.



The branch cuts at positive and negative energy here do not represent a continuum of bound states for negative energy, but rather a continuum of antiparticles (here with a "mass gap").

I leave it as an exercise for you to show that in this system,

$$f(z) = \frac{2z}{z^2+y_0^2} + \frac{1}{\pi}\int_{-\infty}^{\infty} dx' \frac{\mathrm{Im}\, f^+(x')}{x'-z}\,\theta(x'^2-x_0^2) \qquad (34)$$

and to derive this result (as well as the real and imaginary parts of $f^+(x)$) using the 2 contours shown.