SECTION B : PERTURBATIVE METHODS IN Q.M.

As discussed in an accompanying set of "mathematical methods" files, perturbation methods are an integral part of the methods used to solve differential or integral eqtns, or algebraic equations. As such they can be mathematically quite subtle, and so one should view some of the subtleties that arise as peculiar to the the mathematical method rather than to QM itself.

Perturbation theory is thus not the only way to attack problems in QM that are not exactly solvable. Other commonly used methods include asymptotic analysis & related WKB, semiclassical, & multiple-scale methods, & variational methods. And of course one can also attempt expansions in small parameters other than the interaction - the semiclassical and WKB expansion are in powers of $\hbar$, and often makes expansions for small wave numbers (long wavelengths) or small wavelengths. These are just a few of the approximation schemes one uses.

The present section deals with 2 kinds of perturbation. The first applies to the time-independent Schrödinger eqtn., and focusses on expansions about a solvable Hamiltonian, in powers of some parameter connected with the term δH which renders the problem unsolvable. This may be a straightforward expansion in powers of $\delta H/H_0$, where H_0 is the solvable part; or it may be something like ka or $1/ka$, where k is a wavelength and a the range of the perturbation (this perturbation might be a potential $V(r)$, for example). Many of these problems (but by no means all) are scattering problems.

The second kind of perturbation we will examine refers to time-dependent Hamiltonians, where $\mathcal{H} = H_0 + V(t)$. In this case the expansion parameter is either a rate of change $\dot{V}(t)$, divided by a quantity having the dimensions of $(\text{energy})^2$; or the inverse of this. In the first case we have an adiabatic change in the problem, & in the other a sudden change.

A proper study of all this gives us some real bonuses. For example, we will see in a simple way how effective Hamiltonians & renormalisation occurs; we will see how the Born-Oppenheimer approximation can yield a gauge structure to most problems in theoretical physics - and this will lead us to a closer study of quantum phases, including the famous Berry phase. The result is a much more profound understanding of quantum mechanics.

One thing we will not do is look at expansions in powers of $\hbar$. This comes under the heading of semi-classical theory, and will come in section C.

Before beginning, it is useful to understand a little of what kind of exact models & exact results we have - these are the models we will be perturbing around. We therefore begin with these.

B.1. SOME EXACT RESULTS IN Q.M.

There are actually many exact results in Q.M., which can be obtained by more or less sophisticated methods. In this subsection we only look at results obtained using elementary methods, & even then at only a small number of these - the main point is to illustrate methods & to recall some key results. We also eschew tunneling problems here, even though there are plenty of exact solutions - these will be dealt with later on.

In the collection of exact results discussed here we begin with a few well-known results for 1-d potentials, and then go on to deal with centrally symmetric problems in higher dimensions (the central symmetry of course makes them effectively 1-d problems). We will delay all our discussion of problems involving discrete systems (eg, 2-level systems), coupled to each other or to a continuum of states, to later on. We will also deal with simple time-dependent problems later on.

The simplicity of these examples belies their importance - they are the starting point for most of what is done in physics, whether quantum or classical.

B.1.1. PARTICLE IN A POTENTIAL

In this sub-section we deal exclusively with the time-independent Schrodinger eqtn for a single particle, viz.,

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\Psi(\underline{r}) = E\,\Psi(\underline{r}) \qquad (1)$$

which in 1-d reads

$$\boxed{\left[-\frac{\hbar^2}{2m}d_x^2 + V(x)\right]\Psi(x) = E\,\Psi(x) \qquad (1\text{-d})} \qquad (2)$$

In 2 dimension, for a centrally symmetric potential, we separate the wave-fn, writing

$$\Psi(r,\phi) = \psi(r)\,\chi(\phi) \qquad (2d) \qquad (3)$$

so that we have, starting from the 2-d Schrodinger eqtn

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\phi^2\right]\Psi(\underline{r}) = (E - V(r))\,\Psi(\underline{r}) \qquad (4)$$

from which we have

$$\left.\begin{aligned} &(\partial_\phi^2 + \ell^2)\,\chi_\ell(\phi) \\ &\chi_\ell(\phi) \sim \frac{1}{\sqrt{\pi}}e^{i\ell\phi} \end{aligned}\right\} \qquad (5)$$

so that

$$\boxed{\left[\frac{1}{r}\partial_r(r\partial_r) + \frac{2m}{\hbar^2}(E-V(r)) - \ell^2/r^2\right]\psi_\ell(r) = 0} \qquad (2\text{-d}) \qquad (6)$$

which is a 1-d differential eqtn.

In 3 dimensions, we have the separation of variables

$$\left.\begin{aligned}\Psi(r,\theta,\phi) &= \psi(r)\,Y(\theta,\phi) \\ &= \psi(r)\,\Theta(\theta)\,\chi(\phi)\end{aligned}\right\} \qquad (7)$$

and in a standard development in elementary QM, this leads to the results

$$Y(\theta,\phi) \rightarrow Y_{\ell m}(\theta,\phi) = a_m\left(\frac{2\ell+1}{4\pi}\,\frac{(\ell-|m|)!}{(\ell+|m|)!}\right)^{1/2} P_\ell^m(\cos\theta)\,e^{im\phi} \qquad (8)$$

$$\left.\begin{aligned} a_m &= (-1)^m \quad (m>0) \\ &= 1 \quad (m<0)\end{aligned}\right\} \qquad (9)$$

with an associated radial eqn for the function: $\qquad \eta(r) = \frac{1}{r}\psi(r)$ (10)

$$\boxed{\left[\partial_r^2 + \frac{2m}{\hbar^2}(E - V(r)) - \frac{\ell(\ell+1)}{r^2} \right] \eta_\ell(r) = 0} \qquad (3\text{-d}) \qquad (11)$$

which is the 1-d eqtn for a 3-d central symmetric potential.

In terms of the soltns to these eqtns, we see that the general soltn to the 2-d & 3-d problems is

$$\left.\begin{aligned} \Psi(\underline{r}) &= \sum_\ell c_\ell e^{i\ell\phi} \psi_\ell(r) && (2d) \\ \text{and} \quad \Psi(\underline{r}) &= \sum_{\ell m} c_{\ell m} Y_{\ell m}(\theta,\phi)\, r\, \eta_\ell(r) && (3d) \end{aligned}\right\} \qquad (12)$$

where the c_ℓ and the $c_{\ell m}$ are arbitrary coefficients (with the usual normalisation condition).

Many other QM problems for a single particle can be reduced to 1d problems if there is an appropriate symmetry in the problem. However in what follows we will stick to centrally symmetric problem, since our purpose here is primarily to deal with methods.

B.1.1.(a) 1-d POTENTIALS

: One learns a great deal from the solution of a few simple 1-d problems. Even though you've seen at least some of these before, it is useful to have them at hand, and to see what lessons can be learned from them.

Example 1: A δ-Function Potential : This is the simplest problem of all in Q.M. The Schrodinger eqtn (2) reads

$$\left[-\frac{\hbar^2}{2m} d_x^2 + V_0 \delta(x) \right] \psi(x) = E \psi(x) \qquad (13)$$

and we see that we can be dealing with either an attractive potential well or a repulsive barrier. We write (14) in the form

$$\left[\frac{d^2}{dx^2} + (\epsilon - 2g_0\delta(x)) \right] \psi(x) = 0 \qquad \left.\begin{aligned} \epsilon &= \frac{2m}{\hbar^2} E \\ 2g_0 &= \frac{2m}{\hbar^2} V_0 \end{aligned}\right\} \qquad (14)$$

and see how this works in the 2 cases.

(a) $g_0 < 0$ (potential well) : The soltn is clearly made up of plane waves away from the origin; at the origin we have

$$\partial_x \psi \Big|_{x=0^+} - \partial_x \psi \Big|_{x=0^-} = 2g_0 \psi(x=0) \qquad (15)$$

which leads to 1 bound state $|0\rangle = \psi_0$ with energy $E_0 = \hbar^2/2m\, \epsilon_0$, where

$$\epsilon_0 = -|g_0|^2 \qquad \psi_0(x) = (|g_0|/2)^{1/2} e^{-|g_0 x|} \qquad (16)$$

All the positive energy states are found by matching the boundary condition (15) to plane waves - we get the specific solution by adding another boundary condition. Typically this is done by assuming an incoming plane wave,

$$\psi(x) = \begin{cases} \left[e^{ikx} - \dfrac{|g_0|}{|g_0| + ik} e^{-ikx} \right] & x < 0 \\ \dfrac{ik}{|g_0| + ik} e^{ikx} & x > 0 \end{cases} \tag{17}$$

where

$$k^2 = \epsilon_k = \frac{2m}{\hbar^2} E = \frac{1}{\hbar^2} p^2 \tag{18}$$

This result is of course appropriate to a scattering problem. Turning now to the repulsive problem:

(b) $g_0 > 0$ (potential barrier): Now the problem has the same boundary condition, but there are only positive energy solutions, taking the form

$$\psi(x) = \begin{cases} \left[e^{ikx} - \dfrac{g_0}{g_0 - ik} e^{-ikx} \right] & x < 0 \\ \dfrac{-ik}{g_0 - ik} e^{ikx} & x > 0 \end{cases} \tag{19}$$

with k given by the same form as (18)

Thus we see that the general soltn of the problem, for arbitrary g_0, is just (19); with a single bound state described by (16) if $g_0 < 0$.

Later on we shall comment on these results from the point of view of scattering theory. Let us simply comment here that

- The existence of the bound state is inevitable - one can show fairly easily that any potential well in 1-d will give this
- The result in (19) shows that the barrier reflects a part of the incident wave, and transmits the rest. If we calculate the current density associated with these, viz

$$\underline{J}(x) = \frac{-i\hbar}{2m} \left(\psi^*(x) \nabla \psi(x) - \psi(x) \nabla \psi^*(x) \right) \tag{20}$$

we find that these current densities are (when normalised) equal to:

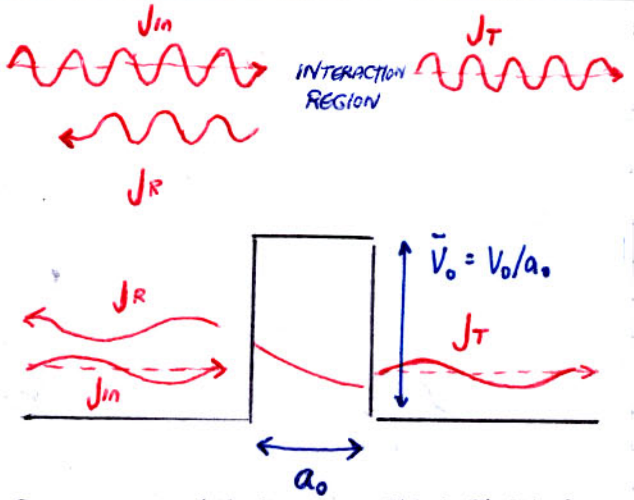
$$\bar{J}_R = \frac{g_0^2}{g_0^2 + k^2} \qquad \bar{J}_T = \frac{k^2}{g_0^2 + k^2} \tag{21}$$

and that their ratio is $\quad J_R / J_T = \tan^2\theta = g_0^2 / k^2 \quad$ (22)

where we define $\quad e^{i\theta} = \dfrac{-ik}{g_0 - ik} \quad$ (23)

Notice the dependence on g_0/k, of all properties of the scattered & reflected waves. We can get further insight into this problem by considering an adapted version in which we choose the

δ-function potential to be the limiting case of a square well/barrier potential. Consider the potential form



PLANE WAVE INCIDENT FROM THE LEFT ON A RECTANGULAR BARRIER, AT 2 ENERGIES

$$\left.\begin{aligned} V(x) &= \frac{V_o}{a_o}\,\theta(a_o^2/4 - x^2) \\ &\xrightarrow[a_o \to 0]{} V_o\,\delta(x) \end{aligned}\right\} \qquad (24)$$

The solution to this can be found by matching the discontinuities in derivatives at $x = \pm a_o/2$ with and incoming plane wave - we do not go through the details. We assume therefore that outside the potential barrier, we can have a solution

$$\left.\begin{aligned} \psi(x) &= A_{in}\,e^{ikx} + A_R\,e^{-ikx} & x < -a_o/2 \\ &= B_T\,e^{ikx} & x > a_o/2 \end{aligned}\right\} \qquad (25)$$

One then finds that the reflected and transmitted current densities are given by

$$\bar{J}_R = \left|\frac{A_R}{A_{in}}\right|^2 = \frac{\frac{1}{4}\bar{V}_o \sin^2\delta(K_o)}{\epsilon_k K_o^2 + \frac{1}{4}\bar{V}_o \sin^2\delta(K_o)} \qquad (26)$$

$$\left.\begin{aligned} K_o^2 &= \frac{2m}{\hbar^2}(E - \bar{V}_o) \\ \epsilon_k &= \frac{2m}{\hbar^2}E \\ \bar{V}_o &= V_o/a_o \\ \delta(K_o) &= K_o a_o \end{aligned}\right\} \qquad (27)$$

$$\bar{J}_T = \left|\frac{B_T}{A_{in}}\right| = 1 - \bar{J}_R = \frac{\epsilon_k K_o^2}{\epsilon_k K_o^2 + \frac{1}{4}\bar{V}_o \sin^2\delta(K_o)} \qquad (28)$$

These results are to be understood as follows for the tunneling case, when $E < \bar{V}_o$; one assumes K_o is imaginary, ie.

$$K_o = i\left[\frac{2m}{\hbar^2}(\bar{V}_o - E)\right]^{1/2} \qquad (E < V_o) \qquad (29)$$

so that, eg.,

$$\bar{J}_R = \frac{\frac{1}{4}\bar{V}_o^2 \sinh^2[(\bar{V}_o - E)^{1/2} a_o]}{E(\bar{V}_o - E) + \frac{1}{4}\bar{V}_o^2 \sinh^2[(\bar{V}_o - E)^{1/2} a_o]} \qquad (30)$$

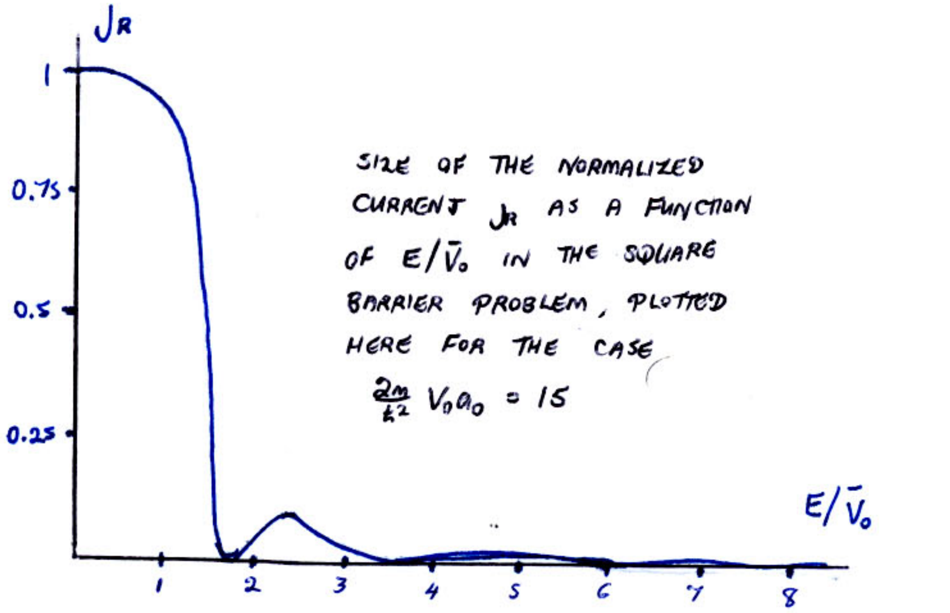
One can also work out formulas for the case where $V_o < 0$ (a potential well). Now let us take a look at these results, which are quite interesting. Note first that the limiting case of the δ-function potential is easily found from (30) (the only applicable result, since in this limit, $\bar{V}_0 \to \infty$ and $a_0 \to 0$); we get

$$\bar{J}_R \xrightarrow[a_0 \to 0]{\bar{V}_0 \to \infty} \frac{\frac{1}{4} V_0^2}{E + \frac{1}{4} V_0^2} \equiv \frac{g_0^2}{k^2 + g_0^2} \qquad (31)$$

Now, for finite a_0, let us look at the behaviour of the reflected current, as a function of the incoming energy. Note that the reflected current is never exactly zero except in the very high energy limit, and also at the values of energy where

$$K_0 a_0 = n\pi \qquad (32)$$

where coherent forward-scattering occurs - this is the well-known optical condition for perfect transmission, first noticed by Hooke & Newton in the 17th century. Nothing in particular happens when $E = \bar{V}_0$; in fact one has that



$$\bar{J}_R = \frac{\frac{1}{4} \bar{V}_0 a_0}{\frac{\hbar^2}{2m} + \frac{1}{4} \bar{V}_0 a_0} \qquad (E = \bar{V}_0) \qquad (33)$$

The limit of low energy is one of total reflection, with a transmission coefficient going to zero exponentially fast - one has

$$\bar{J}_T = 1 - \bar{J}_R \longrightarrow \frac{16 E(\bar{V}_0 - E)}{\bar{V}_0^2} \exp\left\{-2a_0 \left(\frac{2m}{\hbar^2}(\bar{V}_0 - E)\right)^{1/2}\right\} \qquad (K_0 a_0 \gg 1) \quad (34)$$

which is just the tunneling limit.

The importance of these simple results is the way they give simple examples of tunneling, suprabarrier reflection, and resonant transmission, in the case of a potential barrier.

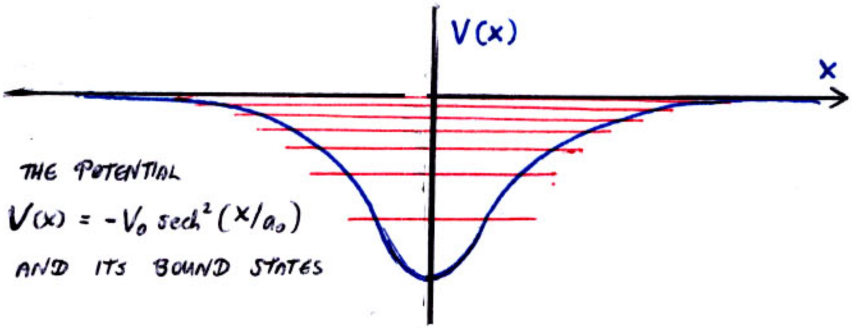
If one works out the simple example of a rectangular potential well, one also finds supra-well reflection and transmission, as well as bound states.

Obviously one can make lots of other simple potentials from piecewise flat portions or sequences of δ-functions - we will look at some of these later on.

<u>Example 2: A sech² Potential Well</u>: We consider this example, even if it is a little complicated in form, because of the relationship to the problem of particles of some field interacting with a soliton of that field. The 1-d Schrodinger eqtn now reads

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - V_0 \,\mathrm{sech}^2(x/a_0)\right]\psi(x) = E\,\psi(x) \qquad (35)$$

This eqtn can be solved by a sequence of substitutions of no fundamental interest to us here. We rescale both $\psi(x)$ and x, as follows; let



THE POTENTIAL
$V(x) = -V_0 \operatorname{sech}^2(x/a_0)$
AND ITS BOUND STATES

$$\psi(x) = \left[\operatorname{sech}\frac{x}{a_0}\right]^{\lambda}\phi(x) \qquad (36)$$

where the constant is

$$\lambda = \frac{1}{2}\left[\left(1+\frac{8mV_0a_0^2}{\hbar^2}\right)^{\frac{1}{2}} - 1\right] \qquad (37)$$

and then rescale the coordinate x according to

$$y = -\sinh^2(x/a_0) \qquad (38)$$

Then one finds a differential eqtn:

$$y(y-1)\,\phi''(y) + \left[(1-\lambda)y - \tfrac{1}{2}\right]\phi'(y) + \left(\lambda^2 - \delta^2(E)\right)\phi(y) = 0 \qquad (39)$$

which is a special case of the hypergeometric eqtn. We do not stop here to discuss the detailed solutions, but simply look at the results for the eigenvalues of the bound states shown above. One finds the eigenenergies E_n, where

$$\left.\begin{aligned} \frac{2m}{\hbar^2}E_n = \epsilon_n &= -(n-\lambda)^2\,\theta(\lambda-n) \\ &\equiv -\theta(\lambda-n)\left[\frac{1}{2}\left(1+\frac{8mV_0a_0^2}{\hbar^2}\right)^{\frac{1}{2}} - \left(n+\tfrac{1}{2}\right)\right]^2 \end{aligned}\right\} \qquad (40)$$

where the Θ-fn indicates that the total number of bound states N in the well is

$$N = 1 + [\lambda] \qquad (41)$$

where $[\lambda]$ is the largest integer less than λ. Thus there is always a bound state, at an energy $\epsilon_0 = -\lambda^2$.

If one considers the free states in this problem and in the previous one, an interesting feature is found - they avoid the potential well. This is because they must be orthogonal to the bound states, so $|\psi_k(x)|^2$ for a high-energy state must be reduced in the region of the potential well. It is interesting to work out the details of this.

B.1.1.(b) LANDAU LEVELS

: A very important example of a problem which, in a certain sense, reduces to a 1-d oscillator problem, is that of a 3-dimensional free particle moving in a uniform and static magnetic field. The Hamiltonian is

$$\left.\begin{aligned} \mathcal{H} &= \frac{1}{2m}\left(\hat{p} + e\underline{A}_0(\underline{r})\right)^2 \\ \text{where}\quad \underline{\nabla}\times A_0(\underline{r}) &= \underline{B}_0 = \hat{z}B_0 \end{aligned}\right\} \qquad (42)$$

To properly specify this problem we need to fix the gauge. It is simplest to use the Landau gauge, viz.

$$A_0(\underline{r}) = (-yB_0, 0, 0) \tag{43}$$

so that

$$\hat{\mathcal{H}} = \frac{1}{2m}(\hat{p}_x - eB_0\hat{y})^2 + \frac{1}{2m}(\hat{p}_y^2 + p_z^2) \tag{44}$$

Now since $\mathcal{H}$ commutes with p_x and p_z (but not $\hat{p}_y$) we can write the eigenfns in the form

$$\psi(\underline{r}) = \phi(y)e^{i(k_x x + k_z z)} \tag{45}$$

where $\phi(y)$ satisfies the eqtn:

$$\phi''(y) - \frac{2m}{\hbar^2}\left[\frac{\hbar^2 k_z^2}{2m} + \tfrac{1}{2}m\omega_c^2(y-y_0)^2 - E\right]\phi(y) = 0 \tag{46}$$

where

$$\omega_c = \frac{|eB_0|}{m} \tag{47}$$

is the cyclotron frequency, and

$$y_0 = -\frac{\hbar k_x}{eB_0} \tag{48}$$

is the mean y-coordinate of the particle. Thus we have a simple harmonic oscillator problem. We can also see this in another way, by mapping the original Hamiltonian to an oscillator. Let us define the canonical momentum

$$\hat{\pi} = \hat{p} + e\hat{A}(\underline{r}) = -i\hbar\nabla_r + eA(\underline{r}) \tag{49}$$

and write the commutation relations between the components as

$$[\hat{\pi}_x, \hat{\pi}_y] = -i\hbar m\omega_c = -i\hbar^2/l_0^2 \tag{50}$$

$$[\pi_z, \pi_y] = [\hat{\pi}_z, \hat{\pi}_x] = 0 \tag{51}$$

where we define a length scale (the "Landau length") as

$$l_0 = (\hbar/|eB_0|)^{1/2} = (\hbar/m\omega_c)^{1/2} \tag{52}$$

and we notice that the flux contained in a circle of radius $2^{1/2} l_0$ is just the flux quantum $\Phi_0 = h/e$, since

$$2\pi B_0 l_0^2 = h/e = \Phi_0 \tag{53}$$

Now we define oscillator ladder operators by taking the usual linear combination of the non-commuting operators in (50), viz.,

$$\left.\begin{aligned} a^+ &= \frac{l_0}{\hbar\sqrt{2}}(\hat{\pi}_x + i\hat{\pi}_y) \\ a &= \frac{l_0}{\hbar\sqrt{2}}(\hat{\pi}_x - i\hat{\pi}_y) \end{aligned}\right\} \tag{54}$$

so that

$$[a, a^+] = 1 \tag{55}$$

Then the Hamiltonian of the system is just

$$\left.\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + \frac{1}{2m}\hat{p}_z^2 \\ \mathcal{H}_0 &= \frac{\hbar\omega_c}{2}(aa^+ + a^+a) = \hbar\omega_c(n+\tfrac{1}{2}) \end{aligned}\right\} \tag{56}$$

From either (45) and (46), or from the mapping to (56), we can then write down the eigenfunctions for the system:

$$\psi_{nk_z}(\underline{r}) = e^{i(k_x x + k_z z)} e^{-\frac{1}{2}m\omega_c(y-y_0)^2} H_n(m\omega_c\sqrt{y-y_0}) \tag{57}$$

where

$$H_n(u) = (-1)^n e^{u^2}\frac{d^n}{du^n}e^{-u^2} \tag{58}$$

is the n-th Hermite polynomial; and from (56) we see that the eigenvalues are

$$E_{nk_z} = \hbar\omega_c(n+\tfrac{1}{2}) + \frac{\hbar^2 k_z^2}{2m} \tag{59}$$

Clearly the eigenvalues are independent of the gauge. However the eigenfunctions are not - and one has a large choice of gauges. Suppose we pick another such gauge - it can always be related to the original gauge by a transformation involving the gradient of a scalar. Let the new gauge be $A'(\underline{r})$. Then we have

$$A'(\underline{r}) = A(\underline{r}) + \nabla f(\underline{r}) \tag{63}$$

and $\psi(r) \to \psi'(r)$, where

$$\psi'(r) = e^{-\frac{i}{\hbar}ef(\underline{r})}\psi(\underline{r}) \tag{64}$$

In particular, it is interesting to go from the Landau gauge to the "symmetric gauge", for which

$$A(\underline{r}) = \tfrac{1}{2}B_0(-y, x, 0) \tag{65}$$

and the derivation of the eigenfunctions in this gauge is left as an exercise.

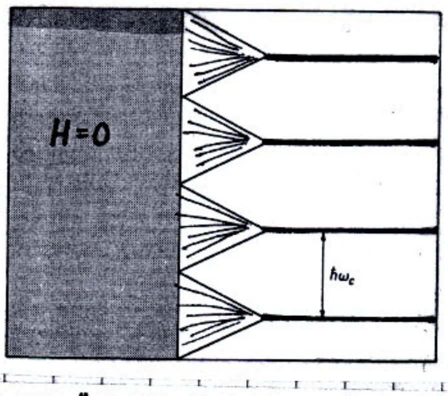
Now let's get some physical feeling for these results, and see why one refers to "Landau levels". Notice first that from (58) or (57), we see that the wave-fns are confined to within a Landau length (in the Landau gauge, this confinement is in the y-direction only, with free motion along $\hat{x}$; in the symmetric or "circular" gauge, it is confined in both directions).

Now imagine that we have a set of N non-interacting fermions with charge e, sitting either in a 2-d plane or in a 3-d box, with sides of length L_x, L_y, and L_z. Ignoring the spin of these particles, we now see that the exclusion principle in momentum space has transformed into an exclusion principle in position space (in the symmetric gauge).

Regardless of which space we count the particles in, the results must be the same - however there is now a big difference, because the energy quantization in (59) imposes a severe restriction on the allowed energy states. This is particularly severe in 2 dimensions, where the $\hbar^2 k_z^2/2m$ disappears.

To see how this works, let's divide the discussion between the 2-d and 3-d cases, and compare the case with Landau quantization to that without. Thus our Hamiltonian is given by

$$\mathcal{H} = \sum_{j=1}^{N} (\underline{p}_j + e\underline{A}(r_j))^2/2m \qquad (66)$$



ABOVE: THE "LANDAU CONDENSATION" OF THE FREE ELECTRONS, IN 2-D, ONTO THE LANDAU LEVELS.
BELOW: THE CORRESPONDING DENSITY OF STATES, A SET OF δ-FUNCTIONS.

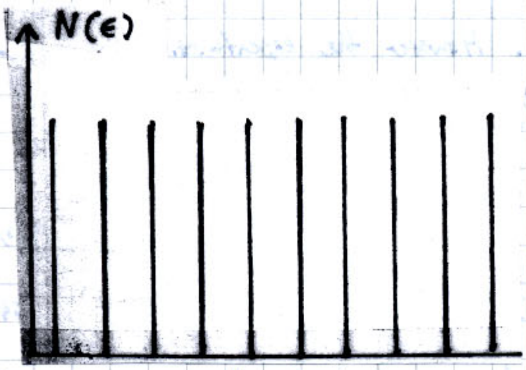
and the result of the Landau quantization is shown at left in schematic form.

<u>2 dimensions</u>: In 2-d, the energy levels allowed to the system are just

$$E_n = \hbar\omega_c (n + \tfrac{1}{2}) \qquad (2d) \qquad (67)$$

Now let's consider what this means for the density of states. When $B_0 = 0$, the density of states for a 2d fermion system, ignoring spin, is just

$$N(\epsilon) = L_x L_y \int \frac{d^2k}{(2\pi\hbar)^2} = L_x L_y \frac{m}{(2\pi\hbar)^2} \qquad (68)$$



for a system with area $L_x L_y$. Now when we apply a field, all states contained in an energy range $\hbar\omega_c$ must condense onto a single Landau level, as shown. The result is then a density of states

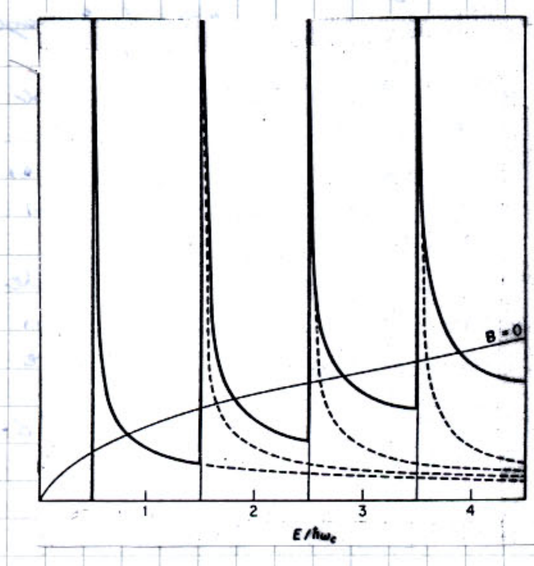
$$\left.\begin{aligned} N(E) &= L_x L_y |B_0| \frac{|e|}{h} \sum_{n=0}^{\infty} \delta(E - E_n) \\ &\equiv \frac{\Phi_{tot}}{\Phi_0} \sum_{n=0}^{\infty} \delta(E - E_n) \end{aligned}\right\} \qquad (69)$$

where $\qquad \Phi_{tot} = |B_0| L_x L_y \qquad (70)$

is the total flux through the system. This result makes it clear that each state on a given level is associated with a single flux quantum, as we might have expected.

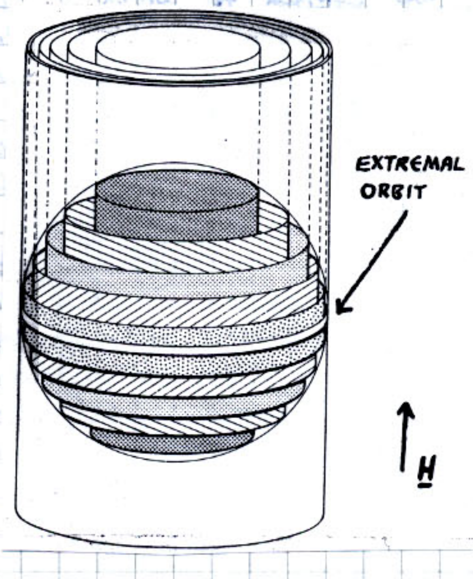
<u>3 dimensions</u>: To get the density of states in 3d, we simply integrate over k_z, using the dispersion relation in (59). The result is

$$N(E) = \frac{L_z}{\pi} \frac{\Phi_{tot}}{\Phi_0} \left(\frac{2m}{\hbar^2}\right)^{1/2} \sum_{n=0}^{\infty} \frac{\theta(E - E_n)}{(E - E_n)^{1/2}} \qquad (71)$$



LEFT: THE DENSITY OF STATES FOR 3d ELECTRONS, FOR $B_0 = 0$ AND FINITE B_0.

RIGHT: THE "LANDAU TUBE" CONSTRUCTION FOR ALLOWED STATES OF FREE ELECTRONS IN 3d MOMENTUM SPACE



The interpretation of this result is interesting. We are essentially adding together a set of states, one set for each Landau level, which if we think of them as being confined to a locus of points in k-space, then look as though they are confined to "tubes" in 3-dimensional k-space, the area of which is given by

$$A_n(k) = \int dk_x\, dk_y\ \theta\left(E_{nk_z} - E^0(\underline{k})\right) \tag{72}$$

where
$$E^0(k) = \frac{\hbar^2}{2m}\left(k_x^2 + k_y^2 + k_z^2\right) \tag{73}$$

This "semiclassical" construction will be re-examined later. The density of states coming from each tube has the $1/\sqrt{E}$ form characteristic of a 1-dimensional Fermi gas: this comes from the integration over k_z. We notice that as $\omega_c \to 0$, the result tends to the form $N(E) \propto E^{1/2}$, characteristic of a 3-d Fermi gas.

One cannot leave a discussion of these densities of states, & the pictures that go with them without remarking on their importance for 20th-century condensed matter physics — much of our early understanding of conductors, starting in 1928, centred on their behaviour in a magnetic field, and the physics of Landau quantization & Landau levels. The presence of these levels has a profound effect on their physical properties (which is completely non-classical).

Incidentally, let us note that in fact Landau was not the first person to analyse or solve this problem (which he looked at in 1930). In fact in 1928 V. Fock solved the more general problem having the Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\left(\underline{p} + eA_0(\underline{r})\right)^2 + \tfrac{1}{2} m \Omega_0^2 r^2 \tag{74}$$

with $\nabla \times A_0(\underline{r}) = \hat{z} B_0$ as before, i.e., a particle moving simultaneously in a magnetic field and a harmonic 3-d potential well. The resulting spectrum is as follows:

$$\left.\begin{aligned} \mathcal{H}\psi_{nmp} &= E_{nmp}\psi_{nmp} \\ E_{nmp} &= \left\{\hbar(\omega_c^2 + \Omega_0^2)^{1/2}(2n + |m| + 1) + m\hbar\omega_c + \hbar\Omega_0(p + \tfrac{1}{2})\right\} \end{aligned}\right\} \tag{75}$$

where
$$\left.\begin{aligned} n &= 0, 1, 2, \ldots \\ m &= 0, \pm 1, \pm 2, \ldots \\ p &= 0, 1, 2, \ldots \end{aligned}\right\} \tag{76}$$

This result has proved very useful in the study of quantum dots.

It is extremely interesting to construct a set of coherent states for this system, and to see what physics can be derived from the use of the 2d-quantized operators defined in (54) above, as well as the coherent states. The points I wish to emphasize in the following are:

(i) the way in which one may extract useful information about the eigenfunctions of the system, and indeed use them, without ever constructing them

(ii) the connection with the angular momentum properties of the system.

(iii) very briefly, the connection to the quantized Hall effect.

Let us start by defining operators
$$c_\alpha = \left(\hat{r}_\alpha + i\frac{\hat{\pi}_\alpha}{m\omega_c}\right) = \left(\hat{r}_\alpha + i\frac{\hat{\pi}_\alpha}{|eB_0|}\right) \tag{77}$$

where $\alpha = x, y$.

The physical significance of these operators is clearer if we write the problem in the symmetric gauge, ie., with a Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\left[\left(\hat{p}_x - \frac{eB_0}{2}\hat{y}\right)^2 + \left(\hat{p}_y + \frac{eB_0}{2}\hat{x}\right)^2 \right] + \frac{p_z^2}{2m} \quad (78)$$

where we have used the gauge $\quad \underline{A}_0(\underline{r}) = \frac{1}{2}B_0(-\hat{y}, \hat{x}, 0) \quad (79)$

If we expand this out it becomes clear that there are 2 conserved quantities in the problem, viz

$$\left.\begin{aligned} x_0 &= \hbar k_y / |eB_0| \\ y_0 &= -\hbar k_x / |eB_0| \end{aligned}\right\} \quad (80)$$

(the quantity y_0 was already given above - see (48)), which are nothing but the centre coordinates of the corresponding classical cyclotron orbit - we will discuss this more later. Thus the operators $\hat{C}_x$, $\hat{C}_y$ are just the

Let us therefore go to a complex variable representation, and define, in addition to the ladder operators a, a^+ in (54), the set of operators

$$\left.\begin{aligned} c &= \frac{1}{l_0\sqrt{2}}(C_x + iC_y) \\ c^+ &= \frac{1}{l_0\sqrt{2}}(C_x - iC_y) \end{aligned}\right\} \quad (81)$$

so that $\quad [c, c^+] = 1 \quad (82)$

So now we have 2 sets of operators, the a, a^+ and c, c^+ operators. Actually they are very similar, as we see if we write them now in terms of the complex variable $x + iy = z$; we have

$$\left.\begin{aligned} a^+ &= \frac{i}{\sqrt{2}}\left(\frac{z}{2l_0} - 2l_0\partial_{z^*}\right) \equiv \frac{i}{\sqrt{2}}\left(\bar{z} - \partial_{\bar{z}^*}\right) \\ a &= \frac{-i}{\sqrt{2}}\left(\frac{z^*}{2l_0} + 2l_0\partial_z\right) \equiv \frac{-i}{\sqrt{2}}\left(\bar{z}^* + \partial_{\bar{z}}\right) \end{aligned}\right\} \quad (83)$$

$$\left.\begin{aligned} c^+ &= \frac{1}{\sqrt{2}}\left(\frac{z^*}{2l_0} - 2l_0\partial_z\right) \equiv \frac{1}{\sqrt{2}}\left(\bar{z}^* - \partial_{\bar{z}}\right) \\ c &= \frac{1}{\sqrt{2}}\left(\frac{z}{2l_0} + 2l_0\partial_{z^*}\right) \equiv \frac{1}{\sqrt{2}}\left(\bar{z} + \partial_{\bar{z}^*}\right) \end{aligned}\right\} \quad (84)$$

where we define the rescaled length $\quad \bar{z} = z/2l_0 = \left(\frac{m\omega_c}{4\hbar}\right)^{\frac{1}{2}} z \quad (85)$

and we recall that $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. Another interesting way of writing these operators is in cylindrical coordinates in the z-plane; noting that if we write $z = re^{i\theta}$, then

$$\partial_x \pm i\partial_y = e^{\pm i\theta}\left(\partial_r \pm \frac{i}{r}\partial_\theta\right) \quad (86)$$

We then find that

$$\left.\begin{matrix} a \\ a^+ \end{matrix}\right\} = \frac{1}{\sqrt{2}} e^{\mp i(\theta + \pi/4)} \left[\left(\bar{r} - \frac{i}{2\bar{r}}\partial_\theta\right) \pm \frac{1}{2}\partial_{\bar{r}}\right]$$

$$\left.\begin{matrix} c \\ c^+ \end{matrix}\right\} = \frac{1}{\sqrt{2}} e^{\pm i\theta} \left[\left(\bar{r} + \frac{i}{2\bar{r}}\partial_\theta\right) \pm \frac{1}{2}\partial_{\bar{r}}\right] \qquad (87)$$

where $\bar{r} = r/2l_0$. We see that these operators act to change the angular momentum and the radial coordinate of the electrons, in a coordinated way.

We can think of the a, a^+ operators as ladder operators for the generalised momenta $\hat{\pi}$, and the c, c^+ operators as those acting on the generalised cyclotron centre coordinate of the electrons. Note that they commute, ie.

$$[a, c] = [a^+, c] = [a^+, c^+] = [a, c^+] = 0 \qquad (88)$$

However they differ greatly in their commutation relations with the Hamiltonian — one has

$$\begin{aligned} [a, \mathcal{H}] &= \hbar\omega_c [a, (a^+a + 1/2)] = \hbar\omega_c a \\ [c, \mathcal{H}] &= 0 \end{aligned} \qquad (89)$$

These relations tell us that the operators a, a^+ move the electrons between Landau levels, with an energy change of $\hbar\omega_c$; whereas the c, c^+ operators leave them in the same Landau level, moving the system between the large number of degenerate states in each Landau level.

We thus need both sets of operators to create/annihilate all the different states in the 2-d system. The "ground state", defined by $b|\Psi_{00}\rangle = a|\Psi_{00}\rangle = 0$, is then given by the analogue of (A.462):

$$\begin{aligned} |\Psi_{00}\rangle &= \frac{1}{\sqrt{2\pi l_0^2}} e^{-zz^*/4l_0^2} \\ &\equiv \frac{1}{\sqrt{2\pi l_0^2}} e^{-\bar{z}\bar{z}^*} \end{aligned} \qquad (90)$$

and the other states in the lowest Landau level can be created by applying c^+ to this vacuum state:

$$\begin{aligned} |\Psi_{0m}\rangle &= (c^+)^m |\Psi_{00}\rangle = \frac{(z^*)^m}{(2^m 2\pi l_0^2 m!)^{1/2}} e^{-zz^*/4l_0^2} \\ \hat{\mathcal{H}}|\Psi_{0m}\rangle &= \tfrac{1}{2}\hbar\omega_c |\Psi_{0m}\rangle \quad \forall\ m \end{aligned} \qquad (91)$$

and the other states are just

$$\begin{aligned} |\Psi_{nm}\rangle &= \frac{(c^+)^m (a^+)^n}{\sqrt{n!\, m!}} |\Psi_{00}\rangle = \frac{i^n (z - 4l_0^2\partial_{z^*})^n (z^*)^m}{(2^{n+m} 2\pi l_0^2 n!\, m!)^{1/2}} \\ \hat{\mathcal{H}}|\Psi_{nm}\rangle &= \hbar\omega_c (n + 1/2) |\Psi_{nm}\rangle \end{aligned} \qquad (92)$$

where we use

$$\left.\begin{array}{llll} \bar{z} = \frac{1}{\sqrt{2}}(c - ia^{+}) & & \partial_{\bar{z}} = \frac{1}{\sqrt{2}}(ia - c^{+}) \\ \bar{z}^{+} = \frac{1}{\sqrt{2}}(c^{+} + ia) & & \partial_{\bar{z}^{+}} = \frac{1}{\sqrt{2}}(c + ia^{+}) \end{array}\right\} \qquad (93)$$

One may now develop the theory in terms of these wave-functions, calculating operator expectation values, etc., in a way similar to that for the 1-d SHO. This is a lot easier than using the position space representation of these wave-fns (which are hypergeometric functions).

Let's very briefly note in passing the connection to the FQHE. Electrons in strong fields in 2d (the so-called 2-d electron gas, or 2DEG), go at low T into a remarkable collective state. In the notation we have been using up to now, this state can be written as

$$\Psi_\nu^{(0)}(\{z_j\}) = \prod_{i<j}(z_i^{+} - z_j^{+})^{1/\nu}\, e^{-\frac{1}{4}\sum_j |z_j|^2} \qquad (94)$$

where we assume (a) that the lowest Landau level is filled up partially, so that a fraction ν of the states in this lowest level are occupied; and (b) that the interactions between the electrons can be treated as short range (in fact the Laughlin wave-fn in (94) is exact in this limit, as the ground state for the system). This state describes electrons orbiting about each other with angular momentum $1/\nu$ (and since ν is a fraction, so is $1/\nu$ - in the original Laughlin theory, ν was equal to $\nu = 1/p$, where p is an integer, so $1/\nu = p$).

The wave function $\Psi_\nu^{(0)}$ in (94) is the ground state - Laughlin also postulated excited states, of the form:

$$\left.\begin{array}{lll} \psi_\nu^{+}(z_0) = \prod_{l=1}^{N}(z_l^{+} - z_0^{+})\, \Psi_\nu^{(0)} & & \text{(hole state)} \\ \psi_\nu^{-}(z_0) = \prod_{l=1}^{N}\left(\partial_{z^{+}} - \frac{z^{+}}{2l_0^2}\right) \Psi_\nu^{(0)} & & \text{(particle state)} \end{array}\right\} \qquad (95)$$

One can develop the theory of coherent states here (compare (95) with eqtn. (A.470), taking $z_0 = 0$). These eqtns describe either a particle or hole state, ie, a state in which a particle has been added or removed from the system.

B.1.2. SOME CENTRAL POTENTIALS

The full theory of scattering will be dealt with in the next couple of sections. Here we simply introduce 2 examples, in order to give us some concrete results. I will assume here you are already familiar with Coulomb scattering from elementary courses.

Scattering theory is just the theory of central potentials, falling to zero sufficiently fast as $r \to \infty$. Here we consider 2 key examples of such problems.

B.1.2.(a) A 2-d SCATTERING PROBLEM

: We are interested in scattering in two dimensions off a short-range potential, which is centrally symmetric. This means that the Schrodinger eqtn reduces to the 2-d radial eqtn (6), along with the 2-d angular harmonics in (5).

The time-independent Schrodinger eqtn in 2d, in the form (6), is put into standard form by substituting the dimensionless variable $x = kr$, where k is defined as before by (18), ie $k^2 = 2mE/\hbar^2$. Then we get

$$\left[\partial_x^2 + \frac{1}{x}\partial_x + (1 - \ell^2/x^2)\right]\psi_\ell(x) = \frac{2mV(x)}{\hbar^2}\psi_\ell(x) \qquad (96)$$

Recall that for arbitrary real μ (ie, not necessarily integer or ½-integer), the solution to the homogeneous differential eqtn

$$\left[d_x^2 + \frac{1}{x}d_x + (1 - \mu^2/x^2)\right]y(x) = 0 \qquad (97)$$

is given by the Bessel functions $J_\mu(x)$, viz.,

$$y(x) = a_1 J_\mu(x) + a_2 J_{-\mu}(x) \qquad (\mu \neq \text{integer}) \qquad (98)$$

However if μ is integer, ie., $\mu \to \ell$, then

$$y(x) = a_1 J_\ell(x) + a_2 Y_\ell(x) \qquad (\ell = \text{integer}) \qquad (99)$$

where $Y_\ell(x)$ is the Neumann function of ℓ-th order.* The properties of these functions are given in many books; we recall here some crucial ones:

<u>General Relations</u>: These take the same form for all the Bessel functions, whether they be Bessel fns of the 1st kind (the $J_\mu(z)$) or of the 2nd kind (the $Y_\mu(z)$). The Neumann fns are defined in terms of the Bessel fns as

$$\left.\begin{aligned} Y_\mu(z) &= \frac{1}{\sin \pi\mu}\left[\cos \pi\mu\, J_\mu(z) - J_{-\mu}(z)\right] \qquad (\mu \neq \ell) \\ &= \lim_{\mu \to \ell} \frac{\cos \pi\mu\, J_\mu(z) - J_{-\mu}(z)}{\sin \pi\mu} = \frac{1}{\pi}\left[\frac{\partial J_\mu(z)}{\partial \mu} - (-1)^\ell \frac{\partial J_{-\mu}(z)}{\partial \mu}\right]_{\mu = \ell} \qquad (\mu = \ell) \end{aligned}\right\} \qquad (100)$$

and both of these fns satisfy the recursion relations (let $X_\mu(z)$ represent $J_\mu(z)$ or $N_\mu(z)$):

$$\left.\begin{aligned} &\frac{d}{dz}\left(z^\mu X_\mu(z)\right) = z^\mu X_{\mu-1}(z) &\qquad &\frac{d}{dz}\left(z^{-\mu}X_\mu(z)\right) = -z^{-\mu}X_{\mu+1}(z) \\ &X_{\mu-1}(z) + X_{\mu+1}(z) = \frac{2\mu}{z}X_\mu(z) &\qquad &X_{\mu-1}(z) - X_{\mu+1}(z) = 2\frac{d}{dz}X_\mu(z) \end{aligned}\right\} \qquad (101)$$

One also defines Hankel functions:

$$H_\mu^\pm(z) = J_\mu(z) \pm i\, Y_\mu(z) \qquad (102)$$

* In the literature the Neumann function is sometimes written as $N_\ell(x)$

(note that the Hankel fns are also called Bessel fns of the 3rd kind, and written as $H^+_\mu(z) = H^{(1)}_\mu(z)$, and $H^-_\mu(z) = H^{(2)}_\mu(z)$); and one can write

$$H^\pm_\mu(z) = \pm \frac{i}{\sin \mu\pi}\left[e^{\mp i\pi\mu} J_\mu(z) - J_{-\mu}(z)\right] \tag{103}$$

All of the 4 functions $J_\mu(z)$, $Y_\mu(z)$, and $H^\pm_\mu(z)$ satisfy the fundamental Bessel eqtn, and any two of them may be used as independent solutns of it. They are often called cylinder functions. To expand functions in terms of them, one uses the orthogonality relations. Suppose we confine ourselves to $\mu > -1$, and assume that the zeroes of $J_\mu(x)$ are at $x = \alpha_n$. Then the orthogonality relation is

$$\int_0^1 dt\, J_\mu(\alpha_n t)\, J_\mu(\alpha_m t) = \frac{1}{2}\delta_{nm}\left[\frac{d}{dt} J_\mu(\alpha_n)\right]^2 \tag{104}$$

If we now want to write the expansion

$$\left.\begin{aligned} f(x) &= \sum_n c_n J_\mu(k_n a) \\ 0 &< x < a \end{aligned}\right\} \tag{105}$$

with k_n defined so that

$$J_\mu(k_n a) = 0 \tag{106}$$

Then we rewrite (104) as

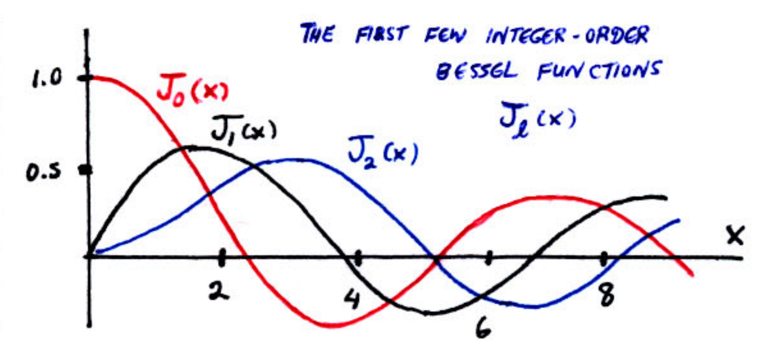
$$\int_0^a x\, dx\, J_\mu(k_n x)\, J_\mu(k_m x) = \frac{a^2}{2}\delta_{nm}\left(J_{\mu+1}(k_n a)\right)^2 \tag{107}$$

using the recursion relations, so that

$$c_n = \frac{2}{a^2}\frac{\int_0^a x\, dx\, J_\mu(k_n x) f(x)}{\left(J_{\mu+1}(k_n a)\right)^2} \tag{108}$$

<u>Form of Bessel functions</u> : It is useful to display these in pictorial form. We begin with the integer-order Bessel functions. These can be calculated in various ways, eg., from the series expansion

$$J_\mu(z) = \sum_{k=0}^\infty \frac{(-1)^k}{k!}\frac{1}{\Gamma(1+k+\mu)}\left(\frac{z}{2}\right)^{2k+\mu} \tag{109}$$



The first 3 Bessel fns are shown at left; their zeroes are at

ℓ \ α_n	1	2	3	4
0	2.404	5.520	8.654	11.792
1	3.832	7.016	10.173	13.323
2	5.135	8.417	11.620	14.796
3	6.379	9.760	13.017	16.224
4	7.586	11.064	14.373	17.616

(110)

For intuitive purposes it is useful to remember that the Bessel functions appear in the solution to the eqtn of motion of a circular membrane. If this has radius a_0, then the soltns for the membrane displacement are

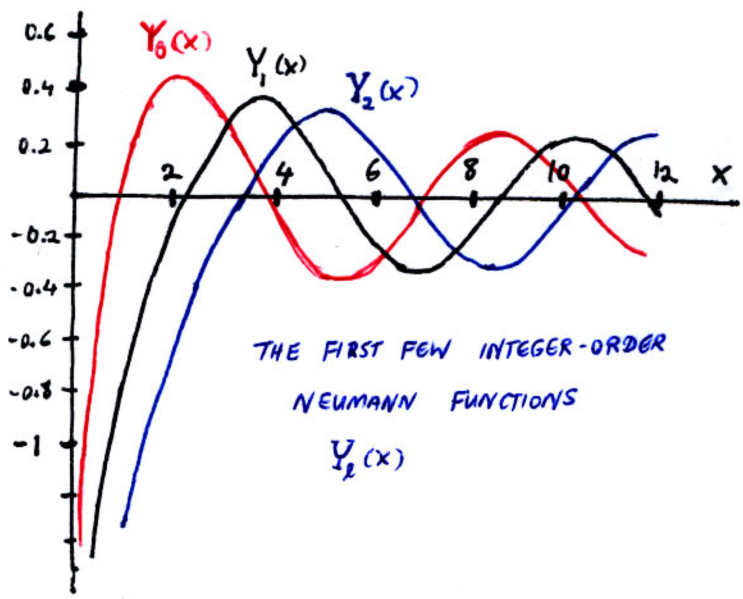
$$u(\phi, r; t) = J_\ell(kr) \begin{Bmatrix} \cos \ell\phi \\ \sin \ell\phi \end{Bmatrix} e^{i\omega t} \qquad (111)$$

with boundary condition $\quad J_\ell(k a_0) = 0 \qquad (112)$

so that the allowed k-values are $\quad k_n = \alpha_n / a_0 \qquad (113)$

It is interesting to draw these solutions.

The Neumann functions diverge at the origin, and so for the simple problem of a free vibrating membrane (or for a free particle solution to Schrödinger's eqtn) they are not required - the first few are shown at left.



THE FIRST FEW INTEGER-ORDER NEUMANN FUNCTIONS $Y_\ell(x)$

However the moment we introduce a potential into the problem, or if we change the boundary conditions, then we need the Neumann functions. Consider, eg., the problem of a 2-d vibrating membrane which now has the ring shape, with the membrane confined so that

$$u(\phi, r; t) = 0 \begin{cases} r > b \\ r < a \end{cases} \qquad (114)$$

Then, since we have 2 boundaries, the soltn necessarily takes the general form (99), ie

$$u(\phi, r; t) = R_\ell(r)\, f_\ell(\phi, t) \qquad (115)$$

where

$$R_\ell(r) = a_\ell J_\ell(kr) + b_\ell Y_\ell(kr) \qquad (116)$$

and the boundary condition (114) becomes

$$\left. \begin{aligned} a_\ell J_\ell(ka) + b_\ell Y_\ell(ka) &= 0 \\ a_\ell J_\ell(kb) + b_\ell Y_\ell(kb) &= 0 \end{aligned} \right\} \qquad (117)$$

Finally, note the asymptotic forms of the Bessel functions:

$$\left. \begin{aligned} J_\mu(z) &\xrightarrow[z\to\infty]{} \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{\pi\mu}{2} - \frac{\pi}{4}\right) \left[1 + O\left(\tfrac{1}{z}\right)\right] \\ Y_\mu(z) &\xrightarrow[z\to\infty]{} \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \frac{\pi\mu}{2} - \frac{\pi}{4}\right) \left[1 + O\left(\tfrac{1}{z}\right)\right] \end{aligned} \right\} \qquad (118)$$

and the curious result that ½-integer Bessel functions reduce to trigonometric fns:

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x \qquad J_{3/2}(x) = \left(\tfrac{2}{\pi x}\right)^{1/2} \left[\tfrac{1}{x} \sin x - \cos x\right] \qquad \text{etc} \ldots \qquad (119)$$
$$J_{-1/2}(x) = \left(\tfrac{2}{\pi x}\right)^{1/2} \cos x \qquad J_{-3/2}(x) = -\left(\tfrac{2}{\pi x}\right)^{1/2} \left[\tfrac{1}{x} \cos x + \sin x\right]$$

Let us now turn to a typical 2-d scattering problem, which is simple and yet demonstrates the basic idea. We imagine a "hard circle" potential of form

$$V(\underline{r}) = \left.\begin{array}{ll} 0 & (r > a_0) \\ \infty & (r \leq a_0) \end{array}\right\} \qquad (120)$$

Now this problem is easily solved using the methods we have just discussed - the boundary condition coming from (120), assuming a solution in the form (116), is just

$$R_\ell(ka_0) = a_\ell J_\ell(ka_0) + b_\ell Y_\ell(ka_0) = 0 \qquad (121)$$

so that

$$a_\ell / b_\ell = - Y_\ell(ka_0) / J_\ell(ka_0) \qquad (122)$$

We can compare this with the problem when there is no potential at all - then one simply has a sum over ordinary Bessel fns, ie., we have components

$$R_\ell^{(0)}(kr) \sim J_\ell(kr) \qquad (123)$$

Let's compare these 2 solutions, by looking at their asymptotic form. From (116) and (122) we have the asymptotic form of $R_\ell(kr)$ as

$$\left.\begin{aligned} R_\ell(kr) \xrightarrow[r\to\infty]{} & \left(\frac{2}{\pi kr}\right)^{1/2} \left[a_\ell \cos\left(kr - \frac{\pi\ell}{2} - \frac{\pi}{4}\right) + b_\ell \sin\left(kr - \frac{\pi\ell}{2} - \frac{\pi}{4}\right)\right] \\ = & \left(\frac{2}{\pi kr}\right)^{1/2} A_\ell \cos\left(kr - \frac{\pi}{2}(\ell + \tfrac{1}{2}) + \delta_\ell\right) \end{aligned}\right\} \qquad (124)$$

where $A_\ell = (a_\ell^2 + b_\ell^2)^{1/2}$, and

$$\left.\begin{aligned} \delta_\ell &= -\tan^{-1}(b_\ell / a_\ell) \\ &= \tan^{-1}\left(\frac{J_\ell(ka_0)}{Y_\ell(ka_0)}\right) \end{aligned}\right\} \qquad (125)$$

As we will see when we do the formal theory of scattering, this phase shift has a fundamental role to play. Notice that we cannot unambiguously define the scattering from a δ-fn potential using this result, because the integral under the hard circle is undefined. To do this, lets consider instead the following potential

$$V(r) = \bar{V}_0 \,\theta(a_0^2 - r^2) = \frac{V_0}{\pi a_0^2}\theta(a_0^2 - r^2) \xrightarrow[a_0 \to 0]{} V_0 \,\delta(\underline{r}) \qquad (126)$$

To deal with problems like this there is a set of standard methods, which we will cover in greater detail in the discussion of scattering theory later on. In the present case the problem is fairly easy to solve, since the boundary conditions at $r = a_0$ are simple. First, the result, for any value of a_0:

$$\delta_\ell = \tan^{-1}\left\{\frac{\beta_\ell(k,\bar{k})\, J_\ell(ka_0) - ka_0 J_\ell'(ka_0)}{\beta_\ell(k,\bar{k})\, Y_\ell(ka_0) - ka_0 Y_\ell'(ka_0)}\right\} \qquad (127)$$

where the function $\beta_\ell(k,\bar{k})$ is

$$\beta_\ell(k,\bar{k}) = \left(\frac{r}{R_\ell(kr)}\frac{dR_\ell(kr)}{dr}\right)\Bigg|_{r=a_0} \qquad (128)$$

and one finds for this problem that, for any value of ka_0:

(i) Above barrier case $(E > \bar{V}_0)$: $$\beta_\ell(k,\tilde{k}) = \tilde{k}a_0 \frac{J_\ell'(\tilde{k}a_0)}{J_\ell(\tilde{k}a_0)} \tag{129}$$

(ii) Below barrier case $(E < \bar{V}_0)$: $$\beta_\ell(k,\tilde{k}) = \tilde{k}a_0 \frac{I_\ell'(\tilde{k}a_0)}{I_\ell(\tilde{k}a_0)} \tag{130}$$

where in the usual way we define $I_\ell(x) = (-i)^\ell J_\ell(ix)$, and

$$k^2 = \frac{2m}{\hbar^2} E \tag{132}$$

and

$$\tilde{k}^2 = \begin{cases} \frac{2m}{\hbar^2}(E - \bar{V}_0) & (E > \bar{V}_0) \\ \frac{2m}{\hbar^2}(\bar{V}_0 - E) & (E < \bar{V}_0) \end{cases} \tag{131}$$

In a minute I will sketch how one gets these results – but let's briefly look at what happens when we take $a_0 \to 0$, to get the δ-fn potential. It is then clear that we are only interested in the below barrier case (130), and that for any finite E we have

$$\left.\begin{aligned} ka_0 &\to 0 \\ \tilde{k}a_0 &= \left(\frac{2mV_0}{\pi}\right)^{1/2} \frac{1}{\hbar} \end{aligned}\right\} (a_0 \to 0) \tag{132}$$

and so for the δ-function potential we get a result INDEPENDENT of V_0:

$$\tan\delta_\ell(k) = \frac{J_\ell(ka_0)}{Y_\ell(ka_0)} \qquad (a_0 \to 0) \tag{133}$$

To analyse this we need the asymptotic properties as $x \to 0$ of the Bessel functions; these are given by

$$J_\ell(x) \xrightarrow[x\to 0]{} \frac{1}{\ell!}\left(\frac{x}{2}\right)^\ell \tag{134}$$

while

$$\left.\begin{aligned} Y_\ell(x) &\xrightarrow[x\to 0]{} \frac{1}{\pi(\ell-1)!}\left(\frac{2}{x}\right)^\ell && (\ell \neq 0) \\ Y_0(x) &\xrightarrow[x\to 0]{} \frac{2}{\pi}\ln\left(C_1 \frac{x}{2}\right) && \ell = 0 \end{aligned}\right\} \tag{135}$$

where $C_1 = e^\gamma$, and γ is the Euler-Mascheroni constant:

$$\left.\begin{aligned} \gamma &= \lim_{s\to\infty}\left(\sum_{k=1}^{s}\frac{1}{k} - \ln s\right) \\ &= 0.577215.. \end{aligned}\right\} \tag{136}$$

(hence $C_1 = 1.781072...$).

Now we see that in the δ-fn limit $a_0 \to 0$, all of the phase shifts go to zero, and in fact we have (PTO):

$$\left.\begin{aligned} \tan\delta_\ell(k) &= \frac{\pi}{\ell}\left(\frac{ka_0}{2}\right)^{2\ell} && (\ell\neq 0) \\ \tan\delta_0(k) &= \frac{\pi/2}{\ln\left(C_1\frac{ka_0}{2}\right)} && (\ell=0) \end{aligned}\right\}(a_0\to 0) \qquad (137)$$

Thus for finite ℓ, the phase shifts decrease very rapidly as a power law in ka_0; however for $\ell=0$ we get a rather slow decrease, going like $1/\ln(ka_0)$, as $ka_0\to 0$.
We shall have cause to look at this result again, in discussing scattering - in fact, even though the phase shift $\delta_0\to 0$ in the δ-fn limit, we shall see that the scattering cross-section in the $\ell=0$ channel actually <u>diverges</u> as $k\to 0$. It turns out that scattering in 2d is full of surprises.

Now let's see how one gets these results. First, notice by comparing (121) and (125), we can write the radial wave-function in the form

$$R_\ell(kr) \propto \left(\cos\delta_\ell(k)\, J_\ell(kr) - \sin\delta_\ell\, Y_\ell(kr)\right) \qquad (138)$$

(we will get an exact expression later on). Now suppose we evaluate the logarithmic derivative of $R_\ell(kr)$ at $r=a_0$, ie., we calculate

$$\beta_\ell = \left(\frac{r}{R_\ell(kr)}\frac{d}{dr}R_\ell(kr)\right)\Big|_{r=a_0} = ka_0\left[\frac{\cos\delta_\ell\, J_\ell'(ka_0) - \sin\delta_\ell\, Y_\ell'(ka_0)}{\cos\delta_\ell\, J_\ell(ka_0) - \sin\delta_\ell\, Y_\ell(ka_0)}\right] \qquad (139)$$

From this we can immediately write $\delta_\ell(k)$ as a function of β_ℓ in the form (127), and it only remains to evaluate $\beta_\ell(k)$. Now this can be done in general by noticing that the logarithmic derivative must be continuous, so that

$$\beta_\ell(r=a_0+\epsilon) = \beta_\ell(r=a_0-\epsilon) \qquad (140)$$

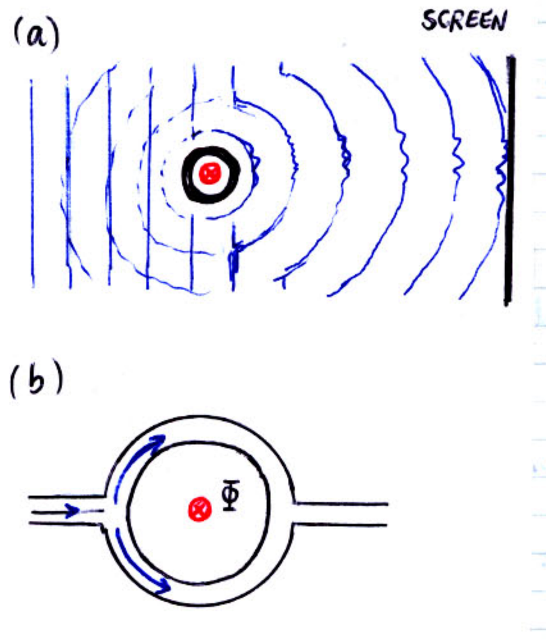
But the solutions for $R_\ell(kr)$ inside the potential $V(r)$ in (126) are obvious - they can only be ordinary Bessel functions $J_\ell(kr)$, since no function like $Y_\ell(kr)$ is allowed (it diverges at the origin). Thus we get the results (129) and (130).
We shall return to problems like this in the section on scattering. Let us now turn to a rather different kind of 2-d scattering problem.

B.1.2. (b) FLUX TUBES and MONOPOLES

: We now turn to a very famous result, which caused considerable disbelief when first published, since it showed that in quantum mechanics (unlike classical mechanics) it is the EM gauge potential $A(r,t)$ that gives physical effects, even in spacetime regions where $\nabla\times A = B$ and $\partial_t A = E$ are zero. This the essential result of Aharonov & Bohm, in 1959.
And yet in a way physicists should not have been surprised. Already in 1931, Dirac had considered the problem of a charged electron moving in the field of a magnetic monopole. At the time the most striking feature of this paper was the result that quantization of the electric charge implied a quantisation of monopole charge - but in retrospect one also sees that these topological effects depend only on the flux that

is enclosed by a given path, and not on the existence of any fields on the paths themselves.

<u>THE AHARONOV-BOHM EFFECT</u>: We consider the set-up shown in (a) in the Figure below. In fact Aharonov & Bohm did not even enclose their infinitesimal flux tube with a "hard circle", which latter can be used to rigorously exclude the electrons from the region where the infinitesimal flux tube penetrates the plane. As they showed (and as we will see) this hard circle does not change the results.



TWO REALISATIONS OF THE AHARONOV-BOHM THOUGHT EXPERIMENT.

The mathematics involved in the Aharonov-Bohm problem is rather tedious, so we will first recall the results for the problem shown in Fig. (b) at left below. In part A this problem was solved completely (cf part A, eqtns (289)-(295)). In particular for a closed ring enclosing a flux ϕ, we found that the eigenfunctions and eigenenergies were

$$\left.\begin{aligned} \psi_\ell(\theta) &= \frac{1}{\sqrt{2\pi}} \exp\{i(\ell+\alpha)\theta\} \\ \epsilon_\ell &= \frac{\hbar^2}{2m}(\ell+\alpha)^2 \end{aligned}\right\} \quad (141)$$

for a ring of unit radius, with electrons of mass m circulating around the ring; and the parameter

$$\alpha = \frac{\Phi}{\Phi_0} = \frac{2\pi e}{\hbar}\Phi = \frac{e}{\hbar}\oint \underline{A}\cdot d\underline{\ell} \quad (142)$$

where e is the charge of the particle, and Φ the flux passing down the flux tube.

Now let's consider the scattering problem shown in Fig. (a) above. We have the usual Hamiltonian

$$\left.\begin{aligned} \mathcal{H} &= \frac{1}{2m}(\underline{p}+e\underline{A}(\underline{r}))^2 \\ \text{with a potential} \quad \underline{A}(\underline{r}) &= \hat{\underline{\theta}}\,\Phi/2\pi r \equiv \Phi\frac{\hat{z}\times\hat{r}}{2\pi r} \equiv \alpha\Phi_0\frac{\hat{z}\times\hat{r}}{2\pi r} \end{aligned}\right\} \quad (143)$$

representing a flux tube of infinitesimal radius, carrying flux Φ.

In the usual way one separates this system into radial & angular components, to get a wave-function

$$\Psi(r,\theta) = \sum_\ell c_\ell \chi_\ell(\theta)\psi_\ell(kr) \equiv \sum_\ell c_\ell \Psi_\ell(\theta,r) \quad (144)$$

satisfying the eqtn:

$$\left\{\partial_r^2 + \frac{1}{r}\partial_r + \left[k^2 - \frac{1}{r^2}(i\partial_\theta+\alpha)^2\right]\right\}\Psi(r,\theta) = 0 \quad (145)$$

From what we have done before on Bessel fns we see that the solution are sums of components of form

$$\Psi(\theta,r) = e^{i\ell\theta} J_{\ell\pm\alpha}(kr) \qquad (r\neq 0) \quad (146)$$

in the region outside the field. However the infinitesimal region inside the field can only have Bessel fns of positive order, so that when we match these (this calculation is properly done using a flux tube of finite radius a_0, and then letting $a_0\to 0$), we get only such terms -

whence we get a solution

$$\Psi(r,\theta) = \sum_{\ell=-\infty}^{\infty} c_\ell e^{i\ell\theta} J_{|\ell+\alpha|}(kr) \qquad (147)$$

To find the coefficients $\{c_\ell\}$ we need to set boundary conditions. It turns out that this is rather a messy problem in general, although we notice that one can write down the Green function for the particle immediately, using (147); we have, for an infinite system

$$\left.\begin{aligned} G(r_2\theta_2; r_1\theta_1; t) &= \sum_\ell \Psi_\ell(r_2,\theta_2)\Psi^*_\ell(r_1\theta_1)e^{-i/\hbar\,\epsilon_\ell(k)t} \\ &= \frac{1}{2\pi}\sum_\ell \int k\,dk\, J_{|\ell+\alpha|}(kr_1)J_{|\ell+\alpha|}(kr_2)e^{i\left[\ell(\theta_2-\theta_1)-\frac{\hbar k^2 t}{2m}\right]} \end{aligned}\right\} \qquad (148)$$

since for the infinite system

$$\epsilon_\ell(k) = \frac{\hbar^2 k^2}{2m} \qquad (149)$$

In their famous paper, Aharonov & Bohm solved the problem for a particle incident in plane wave form, then scattered off to infinity. Since this is a nice example of a scattering problem, we summarise the results here - we will look at it in more detail in the section on scattering.

In a typical scattering problem one assumes a solution of form, in 2 dimensions,

$$\Psi(r,\theta) = e^{ikx} + \frac{1}{\sqrt{r}}f(\theta)e^{ikr} \qquad (r\to\infty) \qquad [2d] \qquad (150)$$

ie., a wave coming in from the left. The form (150) is equivalent to the specification of a boundary condition, which is that a wave has been incident on the scatterer for an infinitely long time, and so that at $r\to\infty$, one sums an incident and scattered wave.

For the Aharonov-Bohm problem, after considerable algebra, one can show that the scattering boundary condition (150) gives a set of coefficients

$$c_\ell = (-1)^{|\ell+\alpha|} \qquad \text{(scattering B.C.)} \qquad (151)$$

and that the actual form of the solution is then, as $r\to\infty$, given by

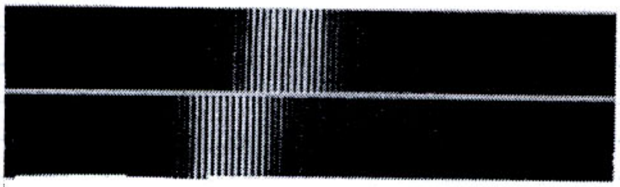
$$\Psi(r,\theta) \sim e^{i(kr\cos\theta+\alpha\theta)} - \frac{e^{ikr+i\pi/4}}{\sqrt{2\pi kr}}\sin\pi\alpha\,\frac{e^{i\theta/2}}{\cos\theta/2} \qquad (152)$$

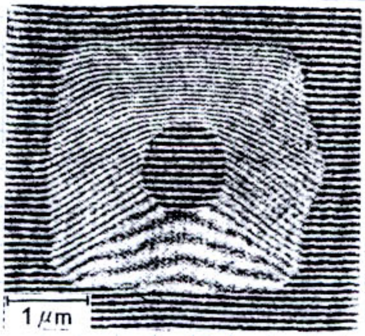
This result shows that the effect of the flux tube is rather complicated - it causes both a phase shift in the incoming plane wave, and it gives a scattered wave whose intensity varies periodically with α, with a maximum when α is ½-integer; and zero if α is an integer.

Actually it turns out that this form breaks down in the direction of forward scattering, and we shall study this in more detail in the section on scattering.

It is interesting at this point to note the way in which the Aharonov-Bohm effect shows up in practise. From (152) we see that one obvious effect of a flux tube will be to shift the fringes in any interference pattern, because of the factor $e^{i\alpha\theta}$ in the plane wave; in fact, at distances such that $kr \ll 1$, the scattered wave will hardly show up in the solution (again, this is not true for forward scattering). From the result for a ring, given in (141), we see that this extra phase factor is nothing but the result of an extra angular momentum $\hbar\alpha$ imparted to the electron by the flux tube, even in the absence of any wave-scattering from the vicinity of the tube. This is one of the more fascinating consequences of the non-local nature of $A(\mathbf{r})$, caused by the long-ranged form $\propto 1/r$.

Some very nice experiments have been done to show how this works in practise. Here I show just a few. A very early experiment looked at the continuous shift of an interference fringe, caused by a flux enclosed in the path of a set of electron waves - this is shown below. Below left we see what happens when electrons are incident on a v. small magnetic ring

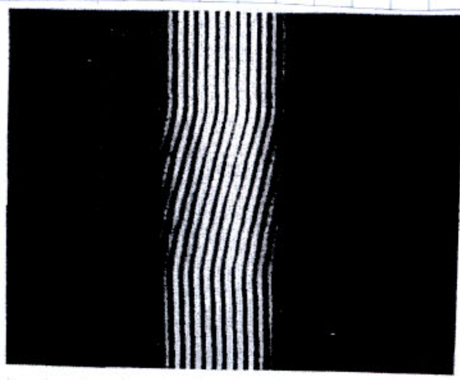




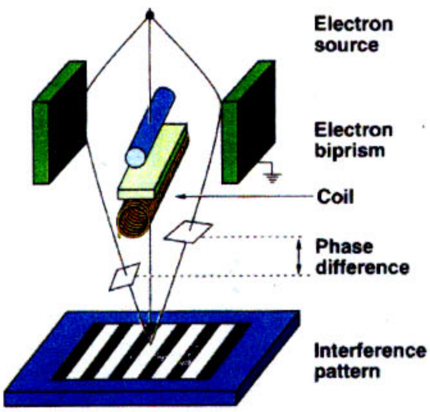
ABOVE: SHIFT OF INTERFERENCE FRINGES IN A 2-SLIT EXPT CAUSED BY INSERTED FLUX.

LEFT: IMAGE FROM ELECTRON WAVES MOVING PAST A TOROIDAL MICROMAGNET.

RIGHT: SHIFTED FRINGES OF ELECTRONS MOVING PAST AN IRON SLIVER

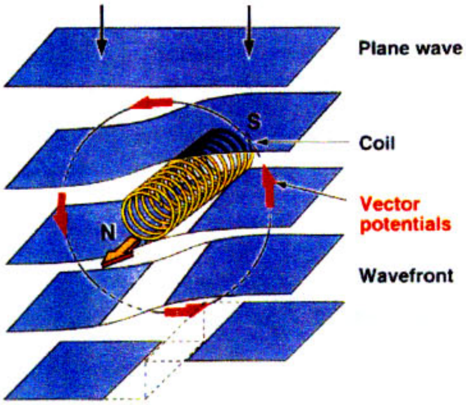


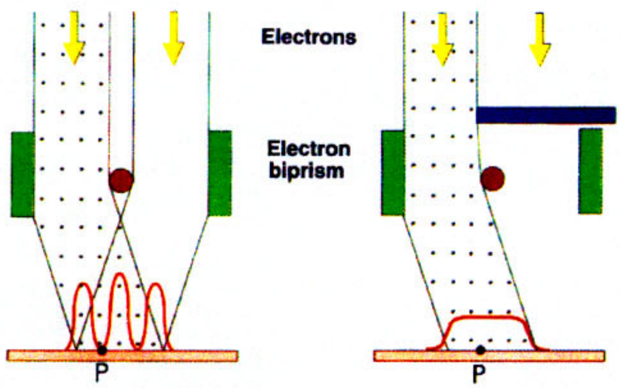
The multiple extra fringes are coming from the flux enclosed within the magnetic ring (inside which flux circulates in a pattern similar to that in a toroidal closed solenoid). The way that each of these experiments work is shown below.



At left we see how an interference experiment of the 2-slit type is done, with the flux solenoid screened from the electron source. The Aharonov-Bohm phase adds to the phase difference between the 2 paths, shifting the fringes. The same set-up is shown below, where we also see how one removes the interference by blocking one of the paths. Nevertheless the other path is deflected by the flux, even though the electrons do not necessarily move through the field.

Finally, below left we see what happens to wave fronts when they pass through a solenoid - one side picks up angular momentum from the circulating $A(\mathbf{r})$, the other loses it. An experiment demonstrating this is shown above.





MAGNETIC MONOPOLE: Now let's turn to a problem which yields fascinating results, interesting both for our understanding of the nature of electric & magnetic charge in particle theory, and for our understanding of spin. Following the remarkable early paper of Dirac, we consider a magnetic monopole of strength g at the origin. The Hamiltonian for an electron is now (adding a central potential):

$$\mathcal{H} = \frac{1}{2m}(\underline{p}+e\underline{A}(\underline{r}))^2 + V(r) \tag{153}$$

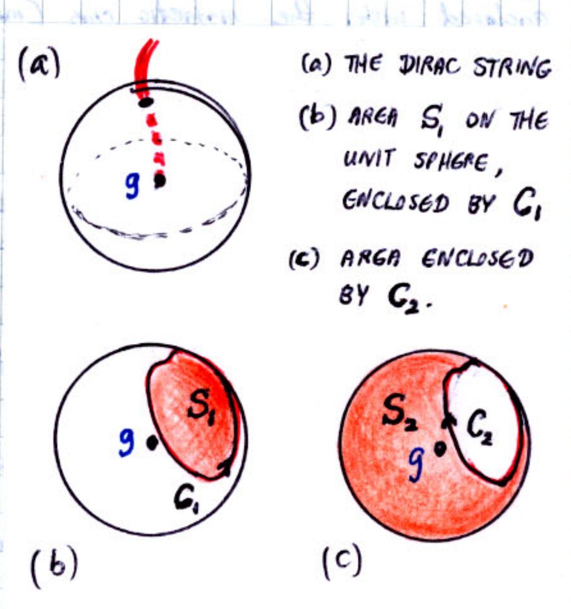
where two common gauge choices are used (we use spherical coordinates (r, θ, ϕ)):

$$\underline{A}^{+}(r) = \left\{ \begin{array}{ll} -\frac{g}{4\pi r}\hat{\phi}\cot\theta/2 & \text{(Polar)} \\ -\frac{g}{4\pi r}\hat{\phi}\frac{1+\cos\theta}{\sin\theta} = \frac{g}{4\pi r}\frac{\hat{\underline{z}}\times\hat{\underline{r}}}{1+\hat{\underline{z}}\cdot\hat{\underline{r}}} & \text{(Dirac)} \end{array} \right\} \tag{154}$$

Both of these give a field:

$$B(r) = \frac{g}{4\pi}\frac{\hat{\underline{r}}}{r} \tag{155}$$

Before looking at the solution to this problem, let's just briefly go over Dirac's famous result relating g and e. The argument is basically very simple & based on single-valuedness of the wave-function. Consider now a circuit made by an electron of charge e around the monopole, at a distance $r=1$, ie., some path on the unit sphere, enclosing a solid angle Ω. Then the total phase accumulated by the electron must be



(a) THE DIRAC STRING
(b) AREA S_1 ON THE UNIT SPHERE, ENCLOSED BY C_1
(c) AREA ENCLOSED BY C_2.

$$\phi(\Omega) = \frac{e}{\hbar}\int_{\Omega}\underline{B}\cdot d\underline{S} = \frac{e}{\hbar}\frac{g}{4\pi}\Omega \tag{156}$$

Now notice that we can define this circuit in 2 ways (see (b) and (c) at left). We can write

$$\phi_1(\Omega) = \frac{e}{\hbar}\oint_{C_1}\underline{A}\cdot d\underline{l} = \frac{e}{\hbar}\int_{\Omega}\underline{B}\cdot d\underline{S} \tag{157}$$

where the circuit C_1 encloses the flux in counter-clockwise sense. However we can also define the circuit as in (c) at left, where the circuit C_2 goes anticlockwise, and encloses not the area S_1 as shown in (b), but its complement S_2; notice that $S_1 = \Omega$, and

$$S_1 + S_2 = 4\pi \tag{158}$$

Now we can calculate ϕ around this curve - we get

$$\phi_2(\Omega) = \frac{e}{\hbar}\oint_{C_2}\underline{A}\cdot d\underline{l} = \left(\frac{e}{\hbar}\int_{\Omega}\underline{B}\cdot d\underline{S} + \frac{e}{\hbar}g\right) \tag{159}$$

where we use (158); the difference between $\phi_1(C)$ and $\phi_2(C)$ is just the flux integrated over the entire sphere. Essentially what we are saying is that the area enclosed by the curve is only defined modulo 4π - whether we are enclosing S_1 or its complement is a matter of choice. Now for the phase to be well-defined, we require $\phi_1(\Omega)$ and $\phi_2(\Omega)$ to be physically equivalent, meaning they must differ by a multiple of 2π, ie

$$\phi_2(\Omega) - \phi_1(\Omega) = ge/\hbar = 2\pi n \qquad \text{ie} \qquad ge = nh \tag{160}$$

This is Dirac's quantization condition - just as in the Aharonov-Bohm effect, it comes from the definition of a phase in terms of a line integral $\int \underline{A}\cdot d\underline{\ell}$; and the requirement that this phase be defined modulo 2π. Ultimately this requirement will be seen to be equivalent to single-valuedness of a wave-function.

Another interesting way to get the same result is by noticing that the vector fields $A(\underline{r})$ in (154) are singular when $\theta \to 0$, ie., along the north pole (in both cases the vector field $\underline{A}(\underline{r})$ circulates along "lines of constant latitude", with magnitude diverging at the north pole, and going to zero at the south pole). Such a divergence in $\underline{A}(\underline{r})$ is inevitable, and follows from an elementary topological theorem (any vector field on a closed simple surface, of genus 0, must have at least one singularity). One imagines a "flux tube", called the Dirac string, coming into the monopole along the positive z-axis, in order to understand the form $\underline{A}(\underline{r})$ in (154). This tube or string carries a flux $\Phi_D = g$, equal to the total flux coming from the real monopole, ie

$$\Phi_D = g = \oint_{4\pi} B.dS \tag{161}$$

However we now notice that we can make a gauge transformation, and change to the following forms for $\underline{A}(\underline{r})$:

$$A^-(\underline{r}) = \left\{ \begin{array}{ll} \hat{\phi}\, \frac{g}{4\pi r} \tan\theta/2 & \text{(anti-Polar)} \\ \hat{\phi}\, \frac{g}{4\pi r} \frac{1-\cos\theta}{\sin\theta} & \text{(anti-Dirac)} \end{array} \right\} \tag{162}$$

Now the Dirac string is coming in along the south pole (and $\underline{A}(\underline{r})$ is circulating in the opposite direction).

Let's now compare the phase change accumulated by a path taken once around the equator, on the unit sphere. If we use the polar or Dirac gauges in (154), we get

$$\phi_+(C) = \frac{e}{\hbar}\oint_{\text{equator}} \underline{A}^+\cdot d\underline{\ell} = e/\hbar \; g/2 \tag{163}$$

whereas if we take the same circuit with $A^-(\underline{r})$ in (162), we get

$$\phi_-(C) = e/\hbar \oint_{\text{equator}} \underline{A}^-\cdot d\underline{\ell} = -e/\hbar \; g/2 \tag{164}$$

Again, we argue that these 2 must be the same modulo 2π, which leads to the conclusion that

$$\phi_+(C) - \phi_-(C) = e/\hbar \; g = 2\pi n \tag{165}$$

leading again to (160), which we also write as:

$$\Phi_D \equiv g = n\Phi_0 \tag{166}$$

ie, the flux carried by the Dirac string must be a multiple of Φ_0.

Thus we arrive at the conclusion that the flux carried by a magnetic monopole must be a multiple of the flux quantum – and in the form (160), that the monopole "charge" g must be inversely proportional to the electric charge e.

With these preliminaries in hand, let's now look at the Schrodinger eqtn for this problem. It is useful to define the operator

$$\underline{J} = \underline{r} \times (p + e\underline{A}(\underline{r})) + \frac{eg}{4\pi}\hat{r} \tag{167}$$

which has components

$$\left.\begin{aligned} \hat{J}_z &= -i\hbar\,\partial_\phi + \frac{eg}{4\pi} \\ \hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y &= e^{\pm i\phi}\left\{ i\hbar \cot\theta\,\partial_\phi \pm \hbar\,\partial_\theta + \frac{eg}{4\pi}\frac{\sin\theta}{1+\cos\theta} \right\} \end{aligned}\right\} \tag{168}$$

where we have used the gauge $\underline{A}^-(\underline{r})$ in (162), in the anti-Dirac form. We notice that the J_α obey the correct commutation relations:

$$[\hat{J}_\alpha, \hat{J}_\beta] = i\hbar\,\epsilon_{\alpha\beta\gamma}\,\hat{J}_\gamma \tag{169}$$

and $\hat{J}^2$ and $\hat{J}_z$ commute with $\mathcal{H}$:

$$[\hat{\mathcal{H}}, \hat{\underline{J}}^2] = [\mathcal{H}, J_z] = 0 \tag{170}$$

Then we can use the "rotational invariance" (under transformations generated by $\hat{\underline{J}}$) to write the Hamiltonian as

$$\mathcal{H} = \frac{1}{2m}\left[P_r^2 + \frac{1}{r^2}\left(J^2 - \frac{e^2g^2}{(4\pi)^2}\right)\right] + V(r) \tag{171}$$

where the radial momentum is

$$P_r = -i\hbar\,\frac{1}{r}\,\partial_r\, r \tag{172}$$

and

$$J^2 = \left\{ -\hbar^2\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2\right) - \frac{2ieg\hbar}{4\pi(1+\cos\theta)}\partial_\phi + \frac{2e^2g^2}{(4\pi)^2(1+\cos\theta)} \right\} \tag{173}$$

Note that the operator $\underline{J}$ is not really an angular momentum – the real angular momentum in the problem is

$$\underline{L} = \underline{r} \times (p + eA(\underline{r})) \tag{174}$$

However L is not conserved, whereas $\underline{J}$ is (and this is true in the equivalent classical problem as well, of course).

To solve for the Hamiltonian in (171) we look for simultaneous eigenfunctions of J^2 and J_z, and write the solution in the form

$$\Psi(r,\theta,\phi) = \psi(r)\,Y(\cos\theta)\,e^{i(m+\mu)\phi} \tag{175}$$

where we define the dimensionless parameter

$$\mu = \frac{eg}{4\pi\hbar} = \frac{n}{2} \tag{176}$$

We will see that the parameter μ plays the same role here as the parameter α in the Aharonov-Bohm problem; however, unlike the Aharonov-Bohm case, μ is quantized, taking integer or ½-integer values, in line with (160). The quantity m is defined by

$$\hat{J}_z \Psi(r,\theta,\phi) = m \Psi(\theta,r,\phi) \qquad (177)$$

and we notice that we must also have

$$\left.\begin{aligned} \hat{J}^2 \Psi(r,\theta,\phi) &= J(J+1)\, \Psi(r,\theta,\phi) \\ \text{and} \qquad m &= J, J-1, \ldots -J. \end{aligned}\right\} \qquad (178)$$

Now, using the separated form in (175), we get a radial eqtn and an eqtn for the angular motion, as follows (defining $\bar{V}(r) = 2mV(r)/\hbar^2$):

$$\left\{ \frac{1}{r}\partial_r^2 r + \left[k^2 - \bar{V}(r) + \frac{1}{r^2}\left(\mu^2 - J(J+1)\right)\right]\right\} \psi(r) = 0 \qquad (179)$$

and, defining $x = \cos\theta$, we have

$$(1-x^2)\, Y''_{(x)} - 2x\, Y'_{(x)} + \left[J(J+1) + \frac{m^2 + 2m\mu x + \mu^2}{x^2-1} \right] Y_{(x)} = 0 \qquad (180)$$

Thus the radial eqtn is in a form we are familiar with, but the angular eqtn is more complicated. I will not go through this here (the detailed discussion can be found in papers in the high-energy literature, going all the way back to Dirac & Tamm in the early 1930's). We write the solution in terms of the MONOPOLE HARMONICS $Y_{Jm\mu}$, defined as

$$\left.\begin{aligned} Y_{Jm\mu}(x) &= C_{Jm\mu}\, (1-x^2)^{m/2} \left(\frac{1-x}{1+x}\right)^{\mu/2} \left(\frac{d}{dx}\right)^{J+m} \left[(1-x^2)^J \left(\frac{1+x}{1-x}\right)^{\mu}\right] \\ \text{where} \qquad C_{Jm\mu} &= \frac{1}{2^J}\left[\frac{(J+\frac{1}{2})(J-m)!}{(J-\mu)!\,(J+\mu)!\,(J+m)!}\right]^{1/2} \end{aligned}\right\} \qquad (181)$$

The energy levels of the system depend on the radial eqtn. and on the boundary conditions we ensure, and can be understood in the same way as in the Coulomb problem, except that the parameter μ now enters everywhere. In the same way one can do scattering theory for the monopole problem.

The monopole problem is interesting not just for particle theorists. We will see in the next main section that it provides us with a way of doing path integrals for spin.

This concludes the discussion of exactly solvable models - it has only touched on the topic, of course, but provided us with a base for doing perturbative approximations, to which we now turn.