A.4. QUANTUM MEASUREMENTS

So far we have said little about the connection between the formalism of QM and the real physical world. The discussion of this is usually referred to by a rather curious phrase, viz, the "interpretation" of Q.M. It is very badly done in most texts, partly because the important conceptual questions are usually glossed over, & also because it is rare to see real examples analysed. The simple fact is that most physicists shy away from this aspect of the theory because within it lurks one of the most important unsolved problems in physics, viz, the problem of reconciling classical ideas about the macroscopic world with the very unclassical nature of the state vector - sometimes called the "measurement problem", for reasons we will come to understand.

In the initial formulation of quantum mechanics the "bridge" between the quantum & classical worlds was provided by the rather mysterious notion of a "measurement" which led to the "collapse of the wave-function". From a modern perspective this is nothing but a rather arbitrary prescription, which raises very awkward questions - & much better discussion is now possible, in which one writes down realistic Hamiltonians which describe the coupling of some macroscopic measuring system to another system which is being measured. The measurement operation then involves a specific kind of entanglement set up between the two, and the "collapse of the wave-function" is a fiction - what actually happens is a loss of coherence in the reduced density matrix of the apparatus because of its entanglement with its "environment" - the same usually also happens to the system being measured. This loss of coherence is usually called "environmental decoherence", and in a quantum measurement it is typically accompanied by a lot of amplification and irreversibility. An analysis of all of this leads ultimately to the Born probabilistic interpretation of QM.

This is a big topic & we can only cover some of it. We begin with an introductory discussion, focussing on the key ideas and a simple example (the Stern-Gerlach system). We then go to a more sophisticated and useful analysis, using the density matrix formalism to good advantage, & introducing a path integral description of density matrix dynamics. This allows a good understanding of the interactions & amplification involved in measurement, and of decoherence. Finally, a few examples are analysed from different fields of physics - this fills out the theoretical discussion in an essential way.

A.4.1. NAIVE MEASUREMENT THEORY

We begin by recalling the standard ideas of wave-function collapse, the correspondence between operators & measurements, & the "von Neumann chain" of measurement operations. We then discuss the usual probabilistic interpretation, & use the example of the Stern-Gerlach expt. to give some idea of what all this is about.

A.4.1.(α) THE VON NEUMANN CHAIN & BORN INTERPRETATION

The earliest discussions of the interpretation of quantum mechs were by the Copenhagen school under Bohr. They were hard to understand, & I will not discuss them until later. It is commonly felt nowadays that the obscurity of the Copenhagen interpretation concealed deep problems in the foundations of the theory.

We begin by considering the 1932 analysis of von Neumann. Suppose we assume a really simple model in which both the quantum system and the measuring

apparatus are in an uncorrelated pure state before they meet. We write the initial wave function of the combined system as

$$\Psi_{in}(S,A) = \Phi_0(A)\,\psi_0(S) \qquad (320)$$

ie., an unentangled product wave-function. This form indicates that the system and apparatus have never met before, and no correlations exist between them.

We then imagine that they interact, in such a way that they do become entangled. In general we can then expect the following transition to occur,

$$\left.\begin{array}{l} \Psi_{in}(A,S) = \sum_\alpha \sum_n a_\alpha c_n |\Phi_\alpha(A)\rangle |\psi_n(S)\rangle \equiv \Phi_0(A)\,\psi_0(S) \quad \text{(BEFORE)} \\ \xrightarrow{\text{interaction}} \quad \Psi_{fin}(A,S) = \sum_\alpha \sum_n F_{\alpha n} |\Phi_\alpha(A)\,\psi_n(S)\rangle \quad \text{(AFTER)} \end{array}\right\} \qquad (321)$$

on to write this in shorthand

$$|\Psi_{in}\rangle = \sum_{\alpha n} a_\alpha c_n |\alpha n\rangle \longrightarrow |\Psi_{fin}\rangle = \sum_{\alpha n} F_{\alpha n} |\alpha n\rangle \qquad (322)$$

where in general $F_{\alpha n}$ cannot be written in product form. Notice that the state afterwards is entangled, even long after interaction has ceased.

Now the transition in (321) & (322) is perfectly general - there is no sign of a measurement. We now define a measurement as follows

$$|\Psi_{in}\rangle = \Phi_0(A) \sum_n c_n \psi_n(S) \longrightarrow \begin{cases} \sum_n c_n \Phi_n \psi_n & \text{(1st kind)} \\ \sum_n c_n \Phi_n \chi_n & \text{(2nd kind)} \end{cases} \qquad (323)$$

where the final state

$$\chi_n(S) = \sum_j U_{jn}\,\psi_j(S) \qquad (324)$$

is quite different from the original one for mmts. of the 2nd kind.

Now the key point here is that the final state of the apparatus is correlated with the initial state of the system - and that in the basis of eigenstates ψ_n, the final state of the apparatus correlates with this.

Thus we may say that if the system starts off in a single state ψ_k, then in the final state, the apparatus must end up in a single state Φ_k. All of this is supposed to be true regardless of the initial state of the apparatus, or of the final state of the system.

The distinction between measurements of the first & 2nd kind (introduced by Pauli) is simply that 1st kind mmts (nowadays sometimes called "Quantum non-Demolition" mmts) leave the system state unchanged, whereas 2nd kind mmts may have a quite different final state for the system.

A MEASURED SYSTEM S & A MEASURING APPARATUS A, BEFORE, DURING, & AFTER INTERACTION $V(S,A)$

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whether or not a measurement is being made:

From the foregoing discussion we see that the kind of interaction that leads to a measurement is really very special - it not leads to a kind of "perfect entanglement between the initial state of S and the final state of A, but also engineers this perfect correlation no matter what is the initial state of A. It is thus clear that the kinds of interaction must be rather special.

Now notice what has happened to the various kinds of density matrix during this transition. Let us rewrite

$$|\Phi_0(A)\rangle \equiv \sum_\alpha a_\alpha \Phi_\alpha(A) \tag{325}$$

as in (321), ie we expand the initial state in a set of states designed to correspond with the states of S.

Now we can say what happens to the various density matrices, as follows:

Mmts of 1st kind	Before	After	
Total density matrix:	$\hat\rho_{in}(S,A) = a_\alpha c_n a_\beta^* c_m^* \lvert\alpha n\rangle\langle\beta m\rvert$	$\hat\rho_{fin} = c_n c_m^* \lvert\Phi_n\psi_n\rangle\langle\Phi_m\psi_m\rvert$	(326)
System " " :	$\bar\rho_{in}(S) = c_n c_m^* \lvert n\rangle\langle m\rvert$	$\bar\rho_{fin}(S) = \sum_n \lvert c_n\rvert^2 \lvert n\rangle\langle n\rvert$	(326)
Apparatus " :	$\bar\rho_{in}(A) = a_\alpha a_\beta^* \lvert\Phi_\alpha\rangle\langle\Phi_\beta\rvert$	$\bar\rho_{fin}(A) = \sum_n \lvert c_n\rvert^2 \lvert\Phi_n\rangle\langle\Phi_n\rvert$	(326)

Mmts of 2nd kind	Before	After	
Total density matrix	$\hat\rho_{in}(S,A) = a_\alpha a_\beta^* c_n c_m^* \lvert\alpha n\rangle\langle\beta m\rvert$	$\hat\rho_{fin} = c_n U_j^n c_m^* U_k^{m*} \lvert\Phi_n\psi_j\rangle\langle\Phi_m\psi_k\rvert$	(327)
System " "	$\bar\rho_{in}(S) = c_n c_m^* \lvert n\rangle\langle m\rvert$	$\bar\rho_{fin}(S) = \sum_n \sum_{jk} \lvert c_n\rvert^2 U_j^n U_k^{n*} \lvert j\rangle\langle k\rvert$	(327)
Apparatus " "	$\bar\rho_{in}(A) = a_\alpha a_\beta^* \lvert\Phi_\alpha\rangle\langle\Phi_\beta\rvert$	$\bar\rho_{fin}(A) = \sum_n \lvert c_n\rvert^2 \lvert\Phi_n\rangle\langle\Phi_n\rvert$	(327)

The key feature to note here is simply that no matter what happens to the system S after the mmt. interaction, the apparatus density matrix after the mmt. is a completely incoherent statistical mixture, with probability $p_n = |c_n|^2$ of "being in state $|\Phi_n\rangle$".

This is one way of setting up the statement of the Born rule for a measuring system, ie., we have

> The probability $p_n = |c_n|^2$ of the measuring system A to be in state $|\Phi_n\rangle$ after the measurement is given directly from the amplitude c_n for the system S to be in the state ψ_n before the measurement. (328)

Note that this means - the probability p_n is defined from the reduced matrix $\bar\rho_{fin}(A)$, provided we only look at the apparatus. It is not actually correct to say that the apparatus *is* in state $|\Phi_n\rangle$ with probability p_n - in reality it is in an entangled state with S. However, as we saw in section A.2 (cf. section A.2.2(b)) we cannot have any information about this if we only look at the apparatus.

Eqtns. (323) provide the first link in what is known as the "von Neumann chain" of mmts. We continue this chain very shortly, but let's ask first - in what sense has a measurement been done? All that has happened so far is that we've put the states of S, in one particular basis, into a rather unusual kind of correlation with those of another system A. Why is this a measurement? To answer this question, let's ask - what sort of properties one might expect a

measurement operation to have? The usual answer given to this question is the following:

(i) A 1-1 correlation between initial state of S and final state of A.
(ii) Some connection between states of A and "macroscopic variables" X.
(iii) Some 1-1 connection between values of X and some quantity associated with the associated state of S } (329)

If these criteria are satisfied, then by reading off the value X_Q of some macroscopic variable associated with the states of A, and with values of some physical quantity Q pertaining to the system S, we can say that we have performed a measurement of Q.

Stage (i) in (329) we have just discussed. What about (ii) and (iii)? Let's first just recall the answer given in the early days of QM, and then discuss the problems it raises.

The first part of the answer asserts that we define the expectation value of some physical quantity Q associated with S by the quantity

$$\langle \hat{A}_Q \rangle = \langle \psi | \hat{A}_Q | \psi \rangle \tag{330}$$

and that if the system wave-function is written in terms of the eigenfunctions $|\psi_n^Q\rangle$ of the operator $\hat{A}_Q$, then

$$\langle A_Q \rangle = \sum_{n\,m} c_n^* c_m \langle \psi_n^Q | \hat{A}_Q | \psi_m^Q \rangle = \sum_n p_n A_n^Q \tag{331}$$

where $p_n = |c_n|^2$ and

$$\hat{A}_Q |\psi_n^Q\rangle = A_n^Q |\psi_n^Q\rangle \tag{332}$$

This is of course what you already know very well. However upon reflection, it tells us very little, even if we already know the operator $\hat{A}_Q$. We still need to know something which is NOT discussed in the usual texts, viz., what is the interaction between S and A which is associated with the operator $\hat{A}_Q$? We will look at this below.

We can now answer (ii) by saying that when the apparatus A is measuring a physical quantity Q associated with S, it must interact with S in the way prescribed by (323), where the $\{|\psi_n\rangle\}$ are eigenstates of the operator $\hat{A}_Q$; and the final states of A are then "reading off" the initial state of S.

The 2nd part of the answer, in the traditional formulation, is very odd – let's run through it. The basic problem in answering (iii) is that the final states described in (326) and (327) are still quantum states. How do we go from these to macroscopic variables, which are assumed to be classical?

The traditional answer appears in various forms, but all of them involve the following replacement

$$\bar{\rho}_{fin}(A) = \sum_n p_n |\Phi_n\rangle\langle\Phi_n| \xrightarrow{\text{collapse}} \text{definite state } |\Phi_n\rangle\langle\Phi_n| \quad (\text{probability } p_n) \tag{333}$$

Moreover, as one sees from (326), if we are dealing with a mixt of the first kind, we have

$$(\text{1st kind})\quad \hat{\rho}_{fin}(S,A) = c_n c_m^* |\Phi_n \psi_n\rangle\langle\Phi_m \psi_m| \xrightarrow{\text{collapse}} \begin{matrix}\text{definite state } n \\ |\Phi_n \psi_n\rangle\langle\Phi_n \psi_n| \\ \text{probability } p_n\end{matrix} \tag{334}$$

This extraordinary transition is called the "collapse of the wave-function". It is of

course not described at all by the Schrödinger eqtn.; in fact, it is not described by any deterministic eqtn. at all. Note what it means - if we look at (333), it says that the system goes from a statistical mixture of states with probability p_n for S, to one definite state - the probability of this being p_n. But (334) is much more dramatic - it is saying that we go from a pure state wave-function, to a statistical mixture, with no apparent mechanism to govern this.

The reason for invoking this desperate measure is that we need to find some way to convert quantum states to classical ones, and to argue that the system finds itself in one of these classical states (ONLY one of them) with probability p_n. The greatest conceptual difficulty with this is of course that not only is the collapse completely arbitrary, but it violates Schrödinger's eqtn - the combined system $S+A$ is assumed isolated.

In the Copenhagen interpretation, the idea that the world was divided into a classical realm and a quantum realm, with the classical realm taken as macroscopic, was taken as axiomatic. This led to a philosophical underpinning for QM which very few have really convincingly claimed to understand. We will examine it later in the course.

Notice finally that in our discussion, we have already assumed the standard Born interpretation of $\psi(r)$, viz, that the probability of finding the system at position r is just

$$P_r = |\psi(r)|^2 = \rho(r,r'). \qquad (335)$$

However if we had not introduced the density matrix formulation at the beginning, we could have introduced this as a postulate, as was done by Born in the beginning of Q.M.

Finally, note that the position of the "cut" between the classical & quantum worlds, ie., the point at which the collapse is supposed to take place, is quite arbitrary in the formalism, provided it takes place somewhere between the observer and the last point at which we expect that quantum correlations, interference, superposition, etc, can be observed. Thus we can follow the chain of entanglement all the way to the observer, only imposing the cut just before it is considered that the mind has taken place.

THE VON NEUMANN ENTANGLEMENT CHAIN TO THE OBSERVER

This then, is the final stage in what we might call the "old-time" orthodox version of QM. The von Neumann chain is written as, schematically:

$$|\Psi_{in}\rangle = |\delta_0; \Phi_0; \ldots \psi^{(2)}; \psi^{(1)}\rangle = |\delta_0; \Phi_0; \ldots \psi_2; \textstyle\sum_n c_n \psi_n\rangle \qquad (336)$$

(Observer, Apparatus, System 2, System 1)

$$\longrightarrow |\delta_0; \Phi_0; \ldots \textstyle\sum_n c_n \psi_n^{(2)} \psi_n^{(1)}\rangle \cdots\rightarrow |\delta_0; \textstyle\sum_n c_n \Phi_n \ldots \psi_n^{(2)} \psi_n^{(1)}\rangle$$

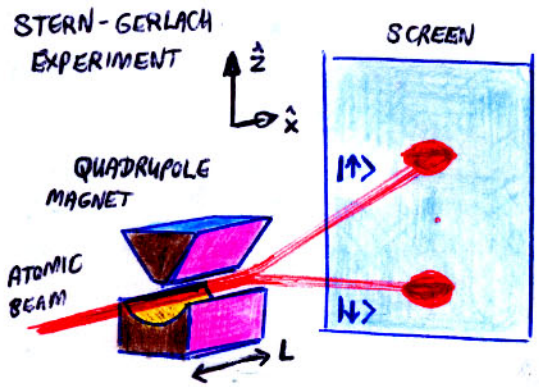
where we have stopped short of questing the observer. At this point von Neumann & others like Peierls imagined that somehow the "consciousness" of the observer intervened to cause the collapse.

All of this smacks very strongly of certain strains of early 20th century continental philosophy, and seems quite curious now - one notices the strong influence of logical positivism (in the emphasis on measurements and observers) as well as various kinds of idealism.

Before turning to a more satisfactory account of matts., let us first fill out the present discussion with a well-known example.

A.4.1.(b) THE STERN-GERLACH EXPERIMENT

: This was one of the first expts to be analysed in any detail in the context of quantum measurement theory. The standard set-up is shown at left - it is a textbook exercise. We assume a beam of neutral spin-½ atoms passes through a quadrupolar field arrangement, and then impinges upon a screen.



Before passing through the magnet, we assume that the spatial & spin coordinates of the atoms are completely uncorrelated, ie., we assume an initial ($t=0$) wave function:

$$\begin{aligned} \Psi_0(\underline{r},\sigma) &\equiv \Phi_0(\underline{r})\,\chi_0(\sigma) \\ &= \Phi_0(\underline{r})\left[a_\uparrow^0|\uparrow\rangle + a_\downarrow^0|\downarrow\rangle\right] \\ &\equiv \left[\Phi_0^\uparrow(\underline{r})|\uparrow\rangle + \Phi_0^\downarrow(\underline{r})|\downarrow\rangle\right] \end{aligned} \qquad (337)$$

where we define the initial coefficients

$$\Phi_0^\sigma(\underline{r}) \equiv \Phi_0(\underline{r})\,a_\sigma^0 \qquad (338)$$

Now consider the interaction inside the quadrupolar field magnet. The key point is that the field is designed to have roughly the form

$$\underline{H}(\underline{r}) \sim \hat{z}\,f(x)\left[H_0 + \alpha z\right] \qquad (339)$$

where $\alpha \equiv \partial H(\underline{r})/\partial z$, and $\underline{H}_0(\underline{r})$ is the inhomogeneous magnetic field from the magnet. The function $f(x)$ is non-zero only inside the magnet, and we assume that

$$\int dx\, f(x) = L \qquad (340)$$

where L is the length of the path through the magnet, during which the spin feels the field in (339). We ignore the effect of fringe fields at the ends of the magnet.

Let's assume that the beam of atoms goes through the magnet at a velocity v_x. Now the Hamiltonian for the system is

$$\begin{aligned} \hat{\mathcal{H}} &= -\mu_e \underline{H}_0(\underline{r})\cdot\underline{\hat{\sigma}} \\ &= -\mu_e f(x)\left(H_0 + \alpha z\right)\hat{\sigma}_z \end{aligned} \qquad (341)$$

The key point here is that there is now a potential in real space, linear in z, whose sign depends on the sign of σ_z. Note however that the uniform part $\underline{H}_0$ of $\underline{H}(\underline{r})$ has no effect on the spatial coordinate of the system.

It is convenient to assume that the beam passes very quickly through the magnet, in a time

$$t_0 = L/v_x \ll 2\pi\hbar/\mu_e H_0 \qquad (342)$$

This condition means that $t_0 \ll$ the time taken for a spin to precess in the field H_0; thus, if a spin oriented along $\hat{x}$, in state $|\rightarrow\rangle$, passes through, it will be unaffected by the field. Thus, as far as the spin is concerned, we are dealing with a mmt. of the first kind.

Now let us consider the action of the Hamiltonian $\hat{\mathcal{H}}(\underline{r})$ in (341), on the motion of the spins. We have

$$\mathcal{H}(\underline{r})\ \Psi^{\sigma}(\underline{r},t) = i\hbar\partial_t \Psi^{\sigma}(\underline{r},t) \qquad (343)$$

where $\Psi^{\sigma}(\underline{r},t) = \Phi^{\sigma}(\underline{r},t)\,|\sigma\rangle \qquad (344)$

and we have the boundary condition $\Psi^{\sigma}(\underline{r},t)\big|_{t=0} = \Psi_0^{\sigma}(\underline{r}) \qquad (345)$

already given in (337). Only the spatial part of the wave-function varies in time because of the condition (342); and we have (assuming the particle enters the magnet at $t=0$):

$$U(t_0) = e^{-\frac{i}{\hbar}\mathcal{H}(\underline{r})t_0} \;\approx\; \exp\left\{-\frac{i}{\hbar}\mu_e\frac{L}{v_x}(H_0+\alpha z)\hat{\sigma}_z\right\} \qquad (t=t_0) \qquad (346)$$

The justification for this result is as follows. The atom passes through the magnet very quickly, and has no time to react to the field while passing through. So we can simply treat the term $\mathcal{H}(\underline{r})t$ in the exponent as $\mathcal{H}(\underline{r})t \to \mathcal{H}(\underline{r})t_0$ (ie., $\mathcal{H}(\underline{r})$ acts during the short time t_0). In time-dependent perturbation theory this is just the "sudden approximation", and it is also, in scattering theory, sometimes called the "impulse approximation".

Now let's consider the effect of (346) on the initial wave-function. To do a proper analysis of this would involve choosing some initial wave-packet, and really allowing the Hamiltonian (341) to act upon it. However here we can get the right answer simply by noting that we can rewrite $U(t_0)$ above as

$$\left.\begin{aligned} U(t_0) &= e^{-\frac{i}{\hbar}\mu_e\frac{L}{v_x}H_0\hat{\sigma}_z}\; e^{-\frac{i}{\hbar}\Delta p_\sigma \cdot z} \\ \text{where}\quad \Delta p_\sigma &= \mu_e\frac{L}{v_x}\alpha\hat{\sigma}_z = \pm\alpha\mu_e\frac{L}{v_x} \end{aligned}\right\} \qquad (347)$$

Now the effect of the first term in (347) is to multiply the spin-up and spin-down wave functions $\Phi^{\sigma}(\underline{r},t)$ by a phase factor, but the second term has a quite different effect - it gives $\Phi^{\sigma}(\underline{r},t)$ a "momentum kick" Δp_σ. The easiest way to see this is to simply apply the momentum operator $p_z = -i\hbar\partial_z$ to the states. We then see that after the beam has passed through the magnet, assuming that the momentum $p_z = 0$ BEFORE going through, ie., that

$$\langle\Phi_0|\hat{p}_z|\Phi_0\rangle = 0 \qquad (348)$$

then we must have $\langle\Phi^{\sigma}(\underline{r},t)|\hat{p}_z|\Phi^{\sigma}(\underline{r},t)\rangle = \Delta p_\sigma \qquad (t \geq t_0) \qquad (349)$

<u>after</u> the atom has passed through. Thus the magnet acts as a beam-splitter, separating spins according to their spin projection along $\hat{z}$.

From the point of view of measurement theory, we can analyse this as follows. The "system" we are interested in is the spin, with wave function $\chi_0(\sigma)$. The "apparatus" is the spatial coordinate z of the atom. The function of the magnetic field is to entangle these 2 coordinates, ie., to make the transition

$$\Psi_0^{\sigma}(\underline{r}) = \Phi_0(\underline{r})\left[a_\uparrow|\uparrow\rangle + a_\downarrow|\downarrow\rangle\right] \longrightarrow \Phi^{\uparrow}(\underline{r},t)|\uparrow\rangle + \Phi^{\downarrow}(\underline{r},t)|\downarrow\rangle \qquad (350)$$

where the spatial wave functions are now $\Phi^{\sigma}(\underline{r},t) \approx \Phi_0(z-\sigma v_z t\,;\, x-v_x t\,;\, y) \qquad (351)$

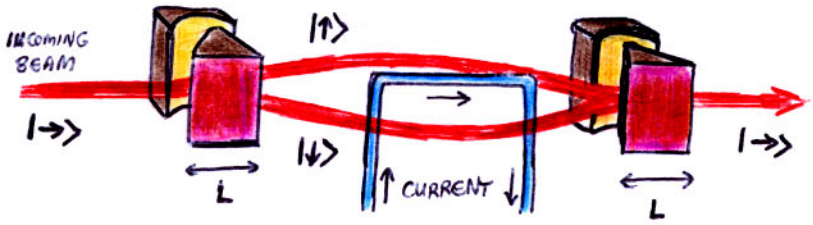
where v_x is the beam velocity as before, and $v_z = \alpha\mu_e L/mv_x \qquad (352)$

using $\sigma v_x = \Delta p^{\sigma}/m$, where m is the mass of the atom.

Of course we do not have to think of the spatial coordinate as the apparatus, but we

see that the wave-packets $\Phi^{\uparrow}$ and $\Phi^{\downarrow}$ are now separating along the z-axis with velocities $\pm v_z$. Notice that if we can recombine these 2 beams with another magnetic field, we do this as shown at left, using another inhomogeneous field (at left, this is generated by an electric current).

RECOMBINATION OF BEAMS IN A STERN-GERLACH SYSTEM



If we think of the spatial coordinate z as the measuring apparatus, then the set-up at left constitutes a reversal of the measurement! For most physicists this seems contradictory, and so they would argue that the measurement has only taken place once it is irreversible. The problem is that whether or not the process is reversible supposedly depends on the ingenuity of the experimentalist; it is obviously very unsatisfactory to have a situation in which the dynamics of the system (in this case the atoms in the beam) is supposed to depend on what the experimentalist might or might not do in the future.

In the present Stern-Gerlach example we would normally say that irreversibility has occured once the particles have hit the screen (thereby entangling the atom spatial coordinate with a huge number of photons, as well as a large number of ions on the screen). But we can still, if we wish, imagine that all of these variables are in a superposition of states.

Notice, however, that there is no way that we would be able to distinguish this "macroscopic superposition" from a mixture unless we were able to perform a measurement on ALL of these variables, using some operator which simultaneously probed them all (cf. discussion in section A.2.2(b)). Thus one often says that this superposition is "for all practical purposes" (FAPP) indistinguishable from a mixture.

A.4.2. QUANTUM MEASUREMENT: REALISTIC THEORY

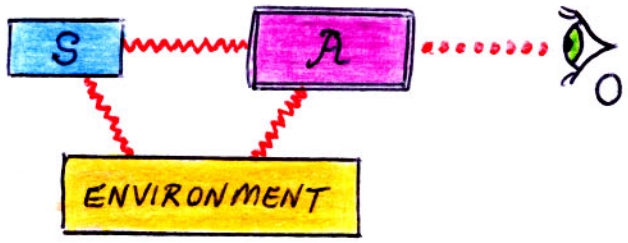
Although the fundamental problem of the interpretation of QM on the macroscopic scale is not yet solved (it may not be without a new theory) we now have a far better understanding of what really happens in nature, which resolves the main problems in the naive interpretation given in the previous subsection. These results & new understanding have all appeared in the last 25 yrs. The key advances are (i) a recognition that a proper understanding of the physics of the measuring system and the environment (ie all other degrees of freedom in the universe) has to be accounted for - these lead to what is called "decoherence" in the dynamics of the measuring apparatus & the measured system, and they may or may not involve amplification; and (ii) accurate and realistic model Hamiltonians which describe this physics.

I will describe some of this physics here, since it not only removes a lot of the mystery associated with measurements, but also gives a good understanding of the physics of entanglement and "disentanglement" (ie, decoherence), and of phase dynamics in quantum systems.

A.4.2(a) REALISTIC MODELS:

One message we can take away from the Stern-Gerlach example is that things really begin to happen once the particles hit the screen - at this point, instead of a few variables being involved, suddenly a huge number of microscopic coordinates become entangled with the spin degree of freedom. So now let us take these seriously, and consider what kind of general models might be used to describe the physics once they are included.

The sort of thing we are looking for is illustrated in schematic form here. We need to couple the degrees of freedom of the rest of the universe to both system and apparatus, as shown. The assumption here is that by the time these 3 systems have finished with each other, the entanglement with the environment is sufficient that the only way we can recover any trace of quantum superposition, interference, etc., in the whole, is to measure all coordinates of the environment. However this is assumed to be impossible - it may include a huge number of degrees of freedom, possibly very far from the system and apparatus.



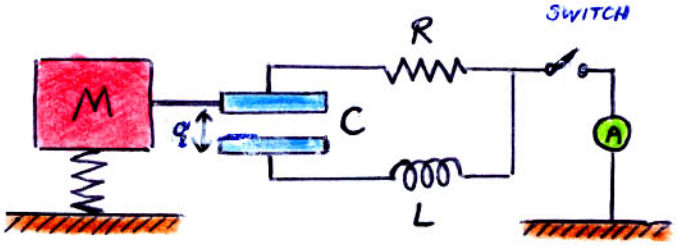
A SYSTEM S COUPED TO AN APPARATUS A; BOTH ARE ALSO COUPLED TO AN ENVIRONMENT. THE OBSERVER O CAN ALSO COUPLE TO A, BUT IS NOT ESSENTIAL.

Notice that in this picture the original superposition still exists - it is now embedded in the wave-function of the entire system (ie., $S + A +$ Environment).

To get an idea of what we are dealing with here, let's look at a few examples. These are chosen to illustrate different aspects of the measuring process, and no attempt is made to be comprehensive.

Example 1: <u>Transducer for Position Measurements</u>: Often in physics one needs to measure the position or the displacement of some object which is confined in some region of space. This object might be microscopic (eg., an atom) or very large (eg., the position of one face of a gravity wave bar detector, or the length of a path in an interferometer). The 2 most common ways of doing this are (i) by converting the position meas. to an electrical meas., using a transducer; and (ii) by using optical detection, & subsequent interference; this is then observed & also converted to an electrical signal using photomultipliers.

Let's take the first example, & comment briefly on the second.



TRANSDUCER DEVICE FOR MEASURING POSITION x OF THE MASS M.

An example is shown at left. We measure the position q of the mass M, by connecting M to one plate of a capacitor in an LRC circuit. If we wish we can do this by measuring the current driven by the motion of the capacitor. Alternatively one could do voltage meas.

The 2 key features of this system to note are (i) that huge numbers of electrons can be made to move by small displacements of M, and electrons are easy to count; and (ii) this measurement process now couples the single coordinate q to a vast number of electrons. It also in principle involves amplification (although no amplifier is shown), and dissipation - no current flow is free of dissipation unless we are in a superconductor (and even there it can).

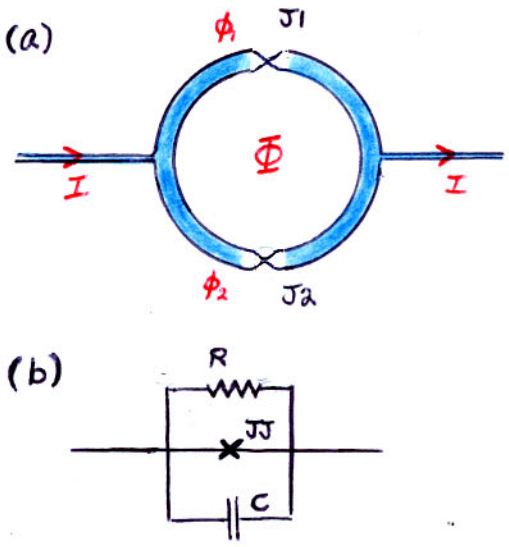
Roughly speaking we can say that the relevant variables are

$$\left.\begin{array}{ll} \text{System } S: & \text{Mass coordinate } q(t) \\ \text{Apparatus } A: & \text{Current } I(t) \quad (\text{collective coordinate of electrons}) \\ \text{Environment } E: & \text{The individual coordinates } \{\mathbf{r}_j; x_j\} \text{ of all the electrons} \\ & (\text{and phonons, impurities, nuclear spins, photons, to which they couple}) \end{array}\right\} \quad (353)$$

Suppose instead we coupled $q(t)$ to a laser field; then $q(t)$ would be measured by interference

in standing photon waves, and to get a reasonable signal, we would amplify this signal to a large electrical one (involving lots of electrons) using a photomultiplier.

Example 2: Mmt of field using a SQUID: Suppose we now want to measure a field $\underline{H}(t)$ which is in general time-dependent. There are many ways of doing this, but let's pick the most sensitive of these. It uses what is called a SQUID (acronym for "Superconducting Quantum Interference Device"). At the core of this is the famous "Josephson Junction", or weak link - typically this is either a narrow constriction or even a break in an otherwise continuous bulk superconductor, or a very small region (the "point contact") where 2 superconductors are forced together. The key point is that electrons can quantum tunnel between the bulk superconductors on either side of the junction. In reality there is always resistive dissipation at a junction, and it will have a small capacitance - hence the "Resistively Shunted Junction" (or RSJ) model shown at left.



(a) THE CENTREPIECE OF A DC SQUID, THE RING WITH 2 JOSEPHSON JUNCTIONS

(b) THE "RSJ" MODEL OF A JOSEPHSON JUNCTION WITH INTRINSIC CAPACITANCE & RESISTANCE

Now, suppose there is some magnetic flux Φ through the ring shown at left, which contains 2 Josephson Junctions (JJ's). Henceforth we will assume that the 2 junctions are identical, and let's assume a ring radius R_0. It then follows that

$$\Phi = \int \underline{A}.d\underline{\ell} = \pi R_0^2 H_0 \qquad (354)$$

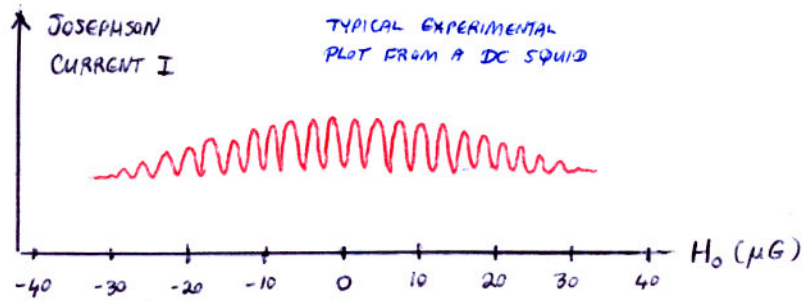
It turns out that the "charges" that carry current in the superconductor have charge $2e$ (they are Cooper pairs) and so the net quantum phase around the ring must be

$$\varphi_c = \frac{1}{\hbar} S_c = 2e \int \underline{A}.d\underline{\ell} = 2\pi n \frac{\Phi}{\Phi_0} \qquad (355)$$

where for the superconducting system with Cooper pair charge $2e$,

$$\Phi_0 = \frac{h}{2e} \qquad (356)$$

is the flux quantum.



A key result, which I will not demonstrate here (it is shown later on) is that the current I through the 2-junction ring shown above is given by (I_0 is a constant):

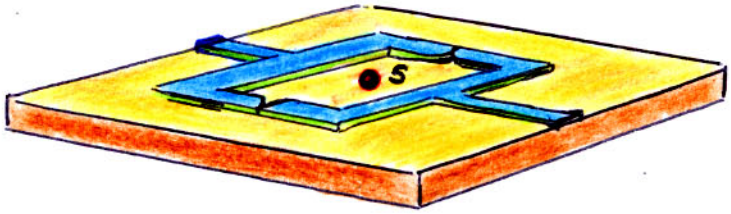
$$I = 2I_0 \cos[\phi_1 + \phi_2] \cos\left(\pi \frac{\Phi}{\Phi_0}\right) \qquad (357)$$

where the phases ϕ_1 and ϕ_2 are the quantum phases across the 2 junctions. In real experiments there is an envelope function as well - this comes from the finite size of the junction. The similarity with 2-slit interference will be obvious, but now we see that the system is an extremely sensitive device for measuring magnetic field; in effect, it counts flux quanta through the ring, and indeed can measure flux to within a small fraction of Φ_0.

We can now consider a typical experimental set-up in which the SQUID is used to detect and measure the field from a very small field source - this could be some current source, or,

as shown in the figure below, a very small magnetic system. This could, eg., be a large spin, which itself is capable of spin dynamics - this dynamics is then detected as a change with time in the flux through the ring.

A SQUID INTERFEROMETER ON AN INSULATING SUBSTRATE, BEING USED TO DETECT THE FIELD FROM A SPIN S INSIDE THE SQUID RING.

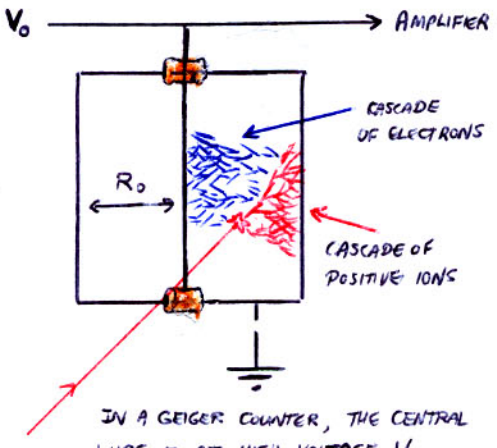


Consider now what degrees of freedom are involved. The spin $\underline{S}$ not only couples to the SQUID current, but also to photons in the EM field, and to nuclear spins $\{\underline{I}_{i\alpha}\}$ in and around it, and also to phonons in and around it. The current $\underline{I}$ in the SQUID will couple to all the incoherent electronic dynamics - this coupling is characterised not only by the resistance R in the RSJ model, but also by entanglement of the current & flux $\underline{\Phi}$ with magnetic impurities & nuclear spins in the SQUID and substrate, and with other "charge defects" in the whole system (these being themselves modelled as two-level systems). Thus we have the following:

System S: the spin $\underline{S}$
Apparatus A: the SQUID current $\underline{I}$, connected to the SQUID flux $\underline{\Phi}$ via (357)
Environment E: the photons, phonons, nuclear spins, paramagnetic spins, defects, and phonons, existing in S, A, and the substrate, & the surroundings. (358)

Notice that in this case, the environment includes variables in the spin, in the SQUID and substrate, in the surrounding vacuum, and in external circuitry, etc - it is everywhere.

Example 3: A Geiger Counter detector: From one point of view a Geiger counter is rather simple - it is a "Yes/No" detector, whose only purpose is to say whether something has happened or not. The basic idea is very simple, and not fundamentally different from the way many modern particle detectors work.



IN A GEIGER COUNTER, THE CENTRAL WIRE IS AT HIGH VOLTAGE V_0, AND THE OUTER WALLS ARE GROUNDED.

A sealed chamber contains a low density gas. The conducting walls are grounded, but at the centre is a wire at very high voltage, V_0. If a high energy particle enters the tube, it will ionize quite a few atoms of gas. The tube is then designed so that the mean free path of electrons in the gas l is such that

$$l/R_0 > E_I/V_0 \qquad (359)$$

where the ionisation energy E_I of the gas atoms is typically $E_I \sim 10\text{-}20\ eV$.

Then it follows that if an atom is ionized, the resulting electron & ion will be accelerated in opposite directions by the field $E \sim V_0/R_0$. If (359) is satisfied, their kinetic energies will be $> E_I$ when they next collide with gas atoms. The result is a massive cascade of charge, which is then absorbed by the walls and by the wire. This current pulse can be amplified, making the detection of a single passing high-energy particle very easy.

This example is shown to demonstrate the massive amplification & irreversibility that

can occur in some measurements. It also illustrates the simple nature of some measurements - here all that is required is that the path $\underline{q}(t)$ of the particle intersect the volume $\Omega(\underline{r})$ of the counter at some time, to get a positive result. Thus we have

System S: The particle trajectory $q(t)$
Apparatus A: The output current $I(t)$ of the counter
Environment E: The coordinates of all gas atoms, charges in the circuitry, etc., & anything else coupling to $I(t)$ or to these charges

(360)

It should be clear now what we are talking about. In the case of Examples 1 and 2 one can quite closely the ideal case of a mmt. of the 1st kind, leaving the system undisturbed by the interaction with A (but not undisturbed by its interaction with E!). However Example 3 is clearly a mmt of the second kind - the incoming particle may be completely destroyed in its interaction with the counter.

Let us now try and distill some sort of general description out of this. Suppose we go back to the scheme caricatured on p. 63, and write an effective Hamiltonian:

$$\mathcal{H} = \mathcal{H}_S^0(q) + \mathcal{H}_A^0(Q) + \mathcal{H}_E^0(X) + V_{int}^{SA}(q,Q) + \mathcal{H}_{int}^{SE}(q,X) + \mathcal{H}_{int}^{AE}(Q,X) \quad (361)$$

where the environmental coordinate X is obviously of various kinds, depending on the example, and typically has very large dimension.

Up until now we have been looking solely at the isolated system/apparatus system defined by the Hamiltonian

$$\mathcal{H}_0^{SA} = \mathcal{H}_S^0(q) + \mathcal{H}_A^0(Q) + V_{int}^{SA}(q,Q) \quad (362)$$

and we have seen what it looks like in a few simple examples (notably the Stern-Gerlach system). What now of the other terms?

This problem has been discussed in great detail in the literature, and the following 2 models are commonly used to deal with 2 kinds of environment - note that both of these models are more appropriate to low energy physics than the kind of system involved in a Geiger counter.

(i) "Oscillator Bath" Models: Suppose the environmental modes that we deal with are actually delocalised, i.e., they can move freely throughout some large system. Then, in a way that is formalised in classical mechanics as the idea of a "normal mode expansion", and in quantum field theory as a free field, one can write the environment Hamiltonian as a set of oscillators:

$$\boxed{\mathcal{H}_E^{OB}(X) = \tfrac{1}{2}\sum_j \frac{p_j^2}{m_j} + m_j\omega_j^2 x_j^2} \quad (363)$$

where the $\{\omega_j\}$ and $\{m_j\}$ are the frequencies and masses of the different oscillators. These modes $\{x_j\}$ are supposed to describe "quasiparticles" like photons, phonons, magnons, etc., which are delocalised throughout some system (or, in the case of photons, throughout all space). One can also use them to describe fermionic electrons at low energies in a conductor (this is not obvious, and will be discussed later on). Notice that the wave-function of these modes will have the form

$$\langle \underline{r} | \psi_j \rangle \sim \frac{x_j}{\sqrt{L}} e^{i\underline{k}_j \cdot \underline{r}} \quad (364)$$

ie., there is a normalising factor $1/\sqrt{L}$, where L is the size of the system, and k_j is some momentum or quasi-momentum associated with the mode, having amplitude (ie, displacement) X_j.

Now how will these oscillators interact with S and A. Here we can use the fact that each mode X_j will typically interact weakly with q and Q, just because they are so spread out in space. This means that usually we can get away with the lowest order (in X_j) coupling, and write

$$\boxed{\begin{aligned} \mathcal{H}_{int}^{SE} &= \sum_j f_j(q,p) X_j + g_j(q,p) P_j \\ \mathcal{H}_{int}^{AE} &= \sum_j F_j(Q,P) X_j + G_j(Q,P) P_j \end{aligned}} \qquad (365)$$

where the couplings are weak, ie., where:

$$f_j, g_j, F_j, G_j \sim O(N_0^{-1/2}) \qquad (366)$$

where $N_0 \sim O(L)$ is the number of oscillator modes in the system. Note that this number depends on the UV cut-off in this Hamiltonian, ie, up to what energy we use the oscillator Hamiltonian (363).

Models like this have been used in a wide variety of physical problems, going back to the early days of both condensed matter & quantum field theory. Later in the course we will see some examples of them, as well as looking at a derivation of them using the Born-Oppenheimer approximation of perturbation theory.

(ii) "Spin Bath" Models: Now imagine that we have a set of localised modes in the environment. This could include phonons or photons or electrons trapped by local potential wells, or defects like interstitials or holes in a solid, or nuclear spins, or paramagnetic impurities, etc. In all of these cases the Hilbert space of these environmental modes is finite - indeed, in the most common version of this model, each environmental mode is a two-level system, which can be mapped to a spin-½ variable $\underline{\sigma}_k$. We can then write

$$\boxed{\mathcal{H}_E^{SB} = \sum_k \underline{h}_k \cdot \underline{\sigma}_k + \sum_{kk'} V_{kk'}^{\alpha\beta} \sigma_k^\alpha \sigma_{k'}^\beta} \qquad (367)$$

where the $\{\underline{h}_k\}$ can be thought of as local "fields" acting on the TLS, and the $V_{kk'}$ are weak interactions between them (weak because the modes are localised and so do not overlap in real space.

The general form of the interactions with S and A is easily written down - we have:

$$\boxed{\begin{aligned} \mathcal{H}_{int}^{SE} &= \sum_k \underline{f}_k(q,p) \cdot \underline{\sigma}_k \\ \mathcal{H}_{int}^{AE} &= \sum_k \underline{F}_k(Q,P) \cdot \underline{\sigma}_k \end{aligned}} \qquad (368)$$

where now the couplings are <u>vector</u> couplings to the $\underline{\sigma}_k$, and there is no particular requirement that these couplings be small - in fact typically $|\underline{F}_k| \gg |V_{kk'}|$.

Note that the forms in (367) and (368) are the most general ones we may write down, as are those in (363) and (365), given the restriction in

(366) and the assumption of a normal mode expansion (which forbids higher order couplings between the oscillators.

Thus we see that these models are very general, and indeed they have been used for many purposes.

In the context of measurement theory the existence of these models completely alters the nature of the discussion, since we may now examine their behaviour and discuss points from a quantitative point of view.

A.4.2(b): DYNAMICS OF QUANTUM MEASUREMENTS

: To get a detailed picture of how measurements we can now start with Hamiltonians like the ones just discussed, and discuss the dynamics of both the total and reduced density matrices for these Hamiltonians. The results turn out to be very interesting. In what follows I merely sketch some of the results, beginning by briefly noting how we deal formally with the dynamics of the reduced density matrices for S and A, and then giving a quick overview of the results. The role of decoherence is stressed, and also the remaining conceptual problems.

DYNAMICS OF REDUCED DENSITY MATRICES

: Let us first recall some essential results of the theory here (a detailed exposition is given in Appendix A). If we have a SINGLE closed system with Hamiltonian $\mathcal{H}$, then assuming that

$$\mathcal{H}|\psi_n\rangle = \epsilon_n|\psi_n\rangle \tag{369}$$

then we can write a propagator K for the density matrix in the form

$$\rho(q_2 q_2'; t_2) = \int dq_1 \int dq_1' \, K(q_2 q_2'; q_1 q_1'; t_2, t_1)\, \rho(q_1 q_1'; t_1) \tag{370}$$

where we have projected onto real space eigenfunctions, and K can be expanded in the system eigenfunctions as

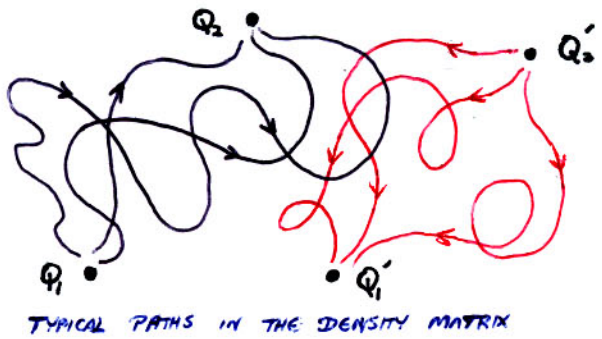
$$\left.\begin{aligned} K(q_2 q_2'; q_1 q_1'; t_2 t_1) &= \sum_{n,m} \psi_n(q_2)\psi_m^*(q_2')\psi_m(q_1')\psi_n^*(q_1)\, e^{-\frac{i}{\hbar}(\epsilon_n - \epsilon_m)(t_2 - t_1)} \\ &\equiv \langle q_2 q_1' | \hat{K}(t_2, t_1) | q_1 q_2' \rangle \end{aligned}\right\} \tag{371}$$

where the operator

$$\hat{K}(t_2, t_1) = \sum_{nm} |nm\rangle\, e^{-\frac{i}{\hbar}(\epsilon_n - \epsilon_m)(t_2 - t_1)} \langle nm| \tag{372}$$

The path integral expression is

$$\left.\begin{aligned} K(q_2 q_2'; q_1 q_1'; t_2 t_1) = \int_{q(t_1)=q_1}^{q(t_2)=q_2} \mathcal{D}q(\tau) \int_{q'(t_1)=q_1'}^{q'(t_2)=q_2'} \mathcal{D}q'(\tau)\; e^{\frac{i}{\hbar}[S[q,\dot{q}] - S[q',\dot{q}']]} \end{aligned}\right\} \tag{373}$$



TYPICAL PATHS IN THE DENSITY MATRIX

and one should take careful note of the positions of the indices, and of the way in which the exponent in (373) involves the difference in the actions of the 2 paths - all of this comes from the definition of the density matrix. From either (371) or (373) we see that this means that the direction of the paths between q_1' and q_2' is opposite to that between q_1 and q_2. Note also that the simplest basis in which to write the

matrix elements of $\hat{K}$ is just the basis of energy eigenstates, in which case we have

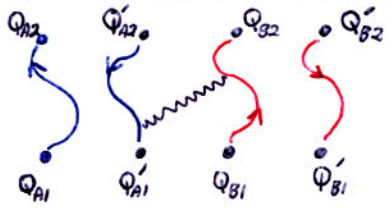
$$K_{nm\,n'm'}(t_2,t_1) = \langle nm|K(t_2,t_1)|n'm'\rangle = e^{-i/\hbar(\epsilon_n-\epsilon_m)(t_2-t_1)}\delta_{nn'}\delta_{mm'} \quad (374)$$

(compare eqn (21Y)).

Now consider the density matrix for a pair of systems A and B. This is clearly a functional of 4 paths. For the general case where they are interacting, we have a Hamiltonian

$$\left.\begin{aligned} \mathcal{H}(A,B) &= \mathcal{H}_0(A) + \mathcal{H}_0(B) + V_{int}(A,B) \\ \text{and an action:}\quad S_{AB}(Q_2^A,Q_1^A;Q_2^B,Q_2^A) &= S_0^A(Q_2^A,Q_1^A) + S_0^B(Q_2^B,Q_1^B) + S_{int}(Q_2^AQ_1^A,Q_2^BQ_1^B) \end{aligned}\right\} (375)$$

and a density matrix matrix propagator



ONE SET OF PATHS FOR THE DENSITY MATRIX FOR SYSTEMS A and B TOGETHER - NOTE THE INTERACTION LINE

$$\left.\begin{aligned} K_{AB}(Q_{A2},Q'_{A2};Q_{B2}Q'_{B2}:Q_{A1},Q'_{A1};Q_{B1}Q'_{B1};t_2,t_1) &= \\ \int_1^2 \mathcal{D}q_A(\tau)\,\mathcal{D}q_B(\tau)\int_1^2\mathcal{D}q'_A(\tau)\,\mathcal{D}q'_B(\tau) & \\ \times \exp\{\tfrac{i}{\hbar}[S_{AB}[q_Aq_B] - S_{AB}[q'_Aq'_B]]\} & \end{aligned}\right\} (376)$$

where we notice that one can also have an interaction between the systems in the path (see Appendices for details).

Suppose we now expand out the difference in actions in (376), ie. write

$$S_{AB}[q_Aq_B] - S_{AB}[q_A,q_B] = (S_0^A[q_A] + S_0^B[q_B] + S_{int}[q_Aq_B]) - (S_0[q'_A] + S_0[q'_B] + S_{int}[q'_Aq'_B]) \quad (377)$$

and now calculate the propagator for the REDUCED density matrix $\bar{\rho}_A$. Then (for a detailed derivation see Appendix A) we have the result (here we suppress the index "A"):

$$\bar{\rho}_A(Q_2Q'_2;t_2) = \int dQ_1\int dQ'_1\ \bar{K}_A(Q_2Q'_2Q_1Q'_1;t_2t_1)\,\bar{\rho}_A(Q_1Q'_1;t_1) \quad (378)$$

where here the propagator $\bar{K}_A$ can in general be dependent on the initial state of the 2nd system (see Appendix A), but we will suppress for the moment. The path integral expression for $\bar{K}_A$ is given by

$$\boxed{\begin{aligned} \bar{K}_A(Q_2Q'_2;Q_1Q'_1;t_2t_1) &= Tr_B\,K_{AB} \\ &= \int_{Q_1}^{Q_2}\mathcal{D}q_A(\tau)\int_{Q'_1}^{Q'_2}\mathcal{D}q'_A(\tau)\,e^{\frac{i}{\hbar}[S_0^A[q]-S_0^A[q']]}\,\mathcal{F}[q,q'] \end{aligned}} \quad (379)$$

where the "influence functional" $\mathcal{F}[q,q']$ is given by tracing the rest of (377), ie.,

$$\boxed{\mathcal{F}[q,q'] = \int dQ_{in}\int dQ_{fin}\int_{Q_{in}}^{Q_{fin}}\mathcal{D}q_B(\tau)\int_{Q_{in}}^{Q_{fin}}\mathcal{D}q'_B(\tau)\,e^{\frac{i}{\hbar}(S_B^0[q_B]-S_B^0[q'_B]+S_{int}[q_Aq_B]-S_{int}[q'_Aq'_B])}} \quad (380)$$

This formula looks very complex but the basic idea is very simple. First notice that

$$\mathcal{F}[q,q'] = 1 \quad \text{when } V_{int}=0 \quad (381)$$

as we would expect - the system is completely disentangled from B. Now consider the role that $F[q,q']$ plays in (379). It is a <u>weighting factor over paths</u>; and its general form is just

$$F[q,q'] = e^{\frac{i}{\hbar}(\Phi[q,q'] + i\Gamma[q,q'])} \tag{382}$$

in which the phase functional $\Phi[q,q']$ describes how the phase of a particle or system at $q(\tau)$ is renormalised by the system at $q'(\tau)$, in the density matrix (and vice-versa), and the "decoherence functional" $\Gamma[q,q']$ shows how the magnitude of $\rho(q,q')$ is diminished by the interaction with the other system, once it has been traced out. Thus $\Phi[q,q']$ and $\Gamma[q,q']$ lead to the renormalisation of both phase and amplitude in $\rho(q,q')$.

A SIMPLE MODEL for MEASUREMENT : Now lets look at the behaviour of the simplest model we can think of for a quantum measurement. In this "PISCES" model (acronym for "pair of interacting spins, coupled to an environmental sea"), we have 2 spin-½ systems coupled to a bath, with Hamiltonian

$$\mathcal{H}_{PISCES} = \underline{B}_1 \cdot \hat{\underline{\sigma}} + \underline{B}_2 \cdot \hat{\underline{\tau}} + V_{int}(\underline{\tau}_1, \underline{\tau}_2) + \mathcal{H}_{Env}(X) + \mathcal{H}^{(1)}_{int}(\hat{\underline{\sigma}}, X) + \mathcal{H}_{int}(\hat{\tau}, X) \tag{383}$$

where for the environment we choose either the oscillator bath form (363) or the spin bath form (364), with the corresponding interaction terms (365) and (368) (note that in the real world the system will couple to both kinds of bath). The interaction V_{int} has the general form

$$V_{int}(\hat{\sigma}, \hat{\tau}; t) = V_{\alpha\beta}(t)\, \hat{\sigma}^\alpha \hat{\tau}^\beta \tag{384}$$

and in general the fields $B_1(t)$ and $B_2(t)$ can also vary in time.

Let's see how this can describe some realistic measurements. Consider first the Stern-Gerlach experiment, where we note that the only important feature is that after the mmt., the sign of the z-coordinate of the centre of mass motion is correlated 1-1 to the spin projection along $\hat{z}$ before the mmt. Thus all details of the spatial form of the wave-fn. are irrelevant, and we can replace it by a spinor, such that the time evolution in the Stern-Gerlach expt. goes like

$$\left.\begin{aligned} |\Psi_{in}\rangle &= |\Rightarrow\rangle_c \left[a|\uparrow\rangle + b|\downarrow\rangle\right]_6 \quad \text{(initial)} \\ |\Psi_{out}\rangle &= a|\Uparrow\uparrow\rangle + b|\Downarrow\downarrow\rangle \quad \text{(final)} \end{aligned}\right\} \tag{385}$$

What sort of interaction Hamiltonian can we use that will accomplish this change? Well, we notice that we need something that will rotate $|\Rightarrow\rangle$ to either $|\Uparrow\rangle$ or $|\Downarrow\rangle$, depending on the sign of σ_z. Noting that the operator $\exp\{2\pi/2\, \hat{\tau}_y\}\, |\tau_z\rangle = -\tau_z|-\tau_z\rangle$, we see that if we choose

$$V_{int}(\hat{\underline{\sigma}}, \hat{\underline{\tau}}) = \left(1 - e^{i\frac{\pi}{2}\hat{\sigma}_z \hat{\tau}_y}\right) \tag{386}$$

then it does the job we want. In reality this interaction will be time-dependent, and we would have something like

$$\left.\begin{aligned} V_{int}(\hat{\sigma}, \hat{\tau}; t) &= \left(1 - e^{i\frac{\pi}{2}\hat{\sigma}_z \hat{\tau}_y f(t)}\right) \\ \int dt\, f(t) &= 1 \end{aligned}\right\} \tag{387}$$

and in the "impulse approximation" we would choose $f(t) \sim \delta(t)$ (with a measurement at $t = 0$).

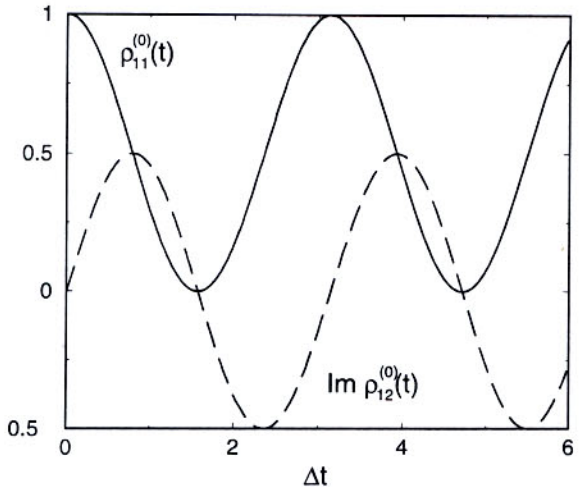
Now consider the role of the environment. Here I give no details of any calculations, which are lengthy, but just the results. Consider first the effect of the environment on a

single one of these spins, uncoupled from the other. Calling this spin $\underline{\tau}$, this means we want to examine the dynamics of either one of the 2 models:

$$\mathcal{H}_{SB} = \Delta_0 \hat{\tau}_x + \epsilon_0 \hat{\tau}_y + \tfrac{1}{2}\sum_j \frac{p_j^2}{m_j} + m_j\omega_j^2 x_j^2 + \sum_j V_j x_j \hat{\tau}_z \qquad \text{(spin-boson)}$$

$$\mathcal{H}_{CS} = \Delta_0 \tau_x + \epsilon_0 \tau_y + \sum_k \underline{h}_k \cdot \underline{\sigma}_k + \sum_{kk'} V_{kk'}^{\alpha\beta} \sigma_k^\alpha \sigma_{k'}^\beta + \hat{\tau}_z \sum_k \underline{V}_k \cdot \underline{\sigma}_k \qquad \text{(central spin)} \qquad (388)$$

These are not the most general forms of these 2 models, but they have been studied very thoroughly and they will do for our purposes. In particular, we know a great deal about the behavior of the reduced density matrix $\tilde{\rho}(\tau_2, \tau_2'; t)$ for these models. Let's see first what the density matrix looks like without any environment, ie., for a "free spin" $\underline{\tau}$. We already know this - it was worked out in section A.3 (see eqtns (299) and (308)). Let's for simplicity assume the bias ϵ_0 is zero in (388). Then the simple result of (299) is shown at left. The free spin shows coherent oscillations when started in an initial state $|\uparrow\rangle$; the frequency of these is of course Δ_0, and the density matrix is then, from (299), given by

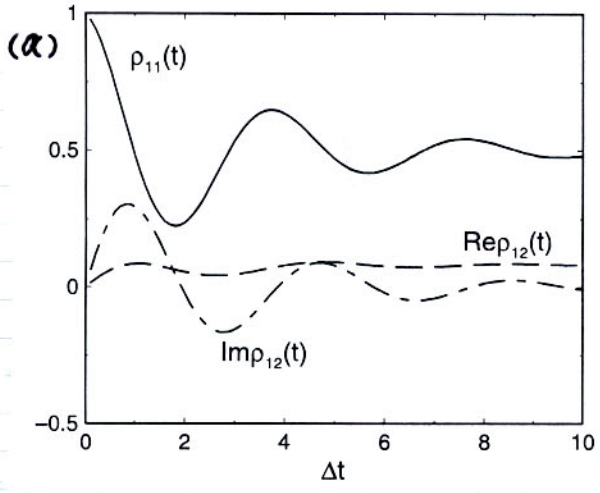


TIME EVOLUTION OF $\rho_{\uparrow\uparrow}(t)$ and $\mathrm{Im}\,\rho_{\uparrow\downarrow}(t)$ FOR THE FREE SPIN, WITH INITIAL STATE $|\uparrow\rangle$

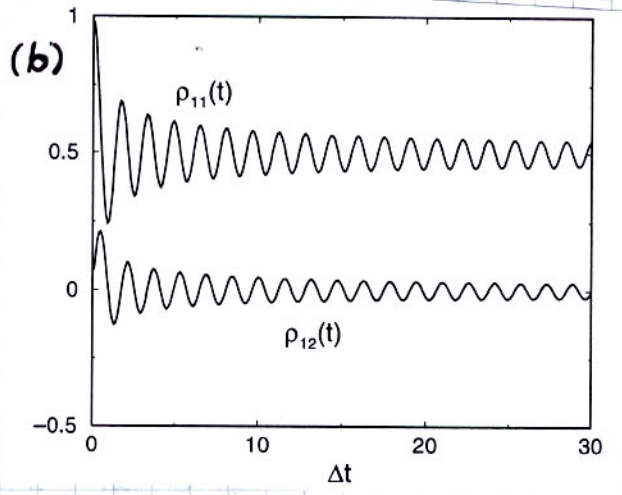
$$\rho_0^{\uparrow\uparrow}(t) = \frac{1}{2}\begin{pmatrix} 1+\cos 2\Delta_0 t & i\sin 2\Delta_0 t \\ -i\sin\Delta_0 t & 1-\cos 2\Delta_0 t \end{pmatrix} \qquad (389)$$

which we see plotted in the figure.

Now consider what happens when we couple to a spin or oscillator bath. It is interesting to contrast these, as we do in the figures below:



(a) TIME EVOLUTION OF REDUCED DENSITY MATRIX OF SPIN ½ COUPLED TO AN OSCILLATOR BATH, WITH INITIAL STATE $|\uparrow\rangle$ & WE ASSUME WEAK COUPLING.



(b) TIME EVOLUTION OF REDUCED DENSITY MATRIX, NOW FOR SPIN COUPLED WEAKLY TO A SPIN BATH (& INITIAL STATE $|\uparrow\rangle$).

Let's say a little bit about these figures. The spin-boson dynamics (a TLS coupled to an oscillator bath) is actually very familiar from many fields. Both the diagonal &

off-diagonal parts of the reduced density matrix show damped oscillations; the characteristic timescales of these oscillations are conventionally known as T_1 and T_2, and it turns out that if the coupling to the bath is weak, we can write

$$\bar{\rho}_{SB}^{\uparrow\uparrow}(t) = \frac{1}{2}\begin{pmatrix} 1+e^{-t/T_1}\cos 2\tilde{\Delta}t & i\,e^{-t/T_2}\sin 2\tilde{\Delta}t \\ -i\,e^{-t/T_2}\sin 2\tilde{\Delta}t & 1-e^{-t/T_1}\cos 2\tilde{\Delta}t \end{pmatrix} \tag{390}$$

where typically $T_1 \gg T_2$ (and in this problem $T_1 \geq 2T_2$ always), and $\tilde{\Delta} < \Delta_0$ is a renormalised frequency.

Physically, T_1 characterises energy relaxation in the system, as the spin $\underline{\tau}$ exchanges energy with the bosonic bath. The shorter time T_2 characterizes DECOHERENCE, the loss of interference between the states $|\uparrow\rangle$ and $|\downarrow\rangle$. When $t \gg T_2$, the density matrix becomes a statistical mixture of $|\uparrow\rangle$ and $|\downarrow\rangle$ states.

The interesting thing about the "central spin" results (where the $\underline{\tau}$-spin couples to spin bath) is that the oscillations do not decay exponentially, but typically as a power law, usually with very odd "resurgences" in the matrix elements (both diagonal & off-diagonal). The important difference between oscillator and spin baths is that the oscillator bath decoherence is always accompanied by energy dissipation - it turns out this is not true for a spin bath.

Now let us return to the "PISCES" problem. We can imagine a situation in which we wish to have the first spin relatively untouched, but have the 2nd apparatus spin relax very quickly. Thus (and this is very common) we would have a very weak coupling to the "system" spin $\underline{\sigma}$, whereas the environment E would couple very strongly to the "apparatus" spin $\underline{\tau}$. What would then be the behaviour.

The answer should by now be obvious. Suppose that we have decoherence timescales T_2^S and T_2^A, with $T_2^A \ll T_2^S$. Nevertheless the coupling between S and A (a coupling like (385) or (387)), which forces entanglement between S and A, will have a dramatic effect. We will find that the joint density matrix will behave as follows:

$$\langle \tau_z \sigma_z | \bar{\rho}_{SA}(t) | \tau_z' \sigma_z' \rangle = \begin{pmatrix} 1+e^{-t/T_1^A}\cos 2(\Delta_S+\Delta_A)t & & & i\,e^{-t/T_2^A}\sin 2(\Delta_S+\Delta_A)t \\ & \text{OTHER MATRIX ELEMENTS} & & \\ -i\,e^{-t/T_2^A}\sin 2(\Delta_S+\Delta_A)t & & & 1-e^{-t/T_1^A}\cos 2(\Delta_S+\Delta_A)t \end{pmatrix} \tag{391}$$

in which the 2 spins oscillate together in the entangled state (the corner matrix elements). Note that all the other matrix elements are not zero - the presence of the environment induces "imperfections" in the correlations between the system and apparatus, which also die away rapidly with time. Readers should compare with the density matrix of (185) in A.2.2 (6), where there is now environment (we have suppressed the phase factor $e^{i\phi}$ here).

We see that because the relaxation time $T_2^A \ll T_2^S$, the decoherence is characterised by the time T_2^A as well.

Now - what does all this mean for the theory of measurements? We see that, if we have no access to the environmental degrees of freedom, the combined system + apparatus density matrix shows a very rapid collapse to an incoherent

density matrix form, in a time T_2^A which is controlled by the coupling between the apparatus A and the environment.

It is in this sense that one can say that FAPP ("For All Practical Purposes") there is no difference between the state of the combined system S+A and a completely classical mixture; after a time T_2^A, all interference between the 2 branches of the wave-function, $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, is FAPP unobservable, because no operator acting on S and A simultaneously can distinguish the pure state from a mixture after this time.

However, we caution that for many physicists, this is not good enough. This is because the interference terms and the superposition are still "out there", embedded in a wave-function of form

$$|\Psi_{UNIV}\rangle = a_\uparrow |\uparrow\uparrow\rangle |E_\uparrow\rangle + a_\downarrow |\downarrow\downarrow\rangle |E_\downarrow\rangle \qquad (292)$$

where $|E_6\rangle$ is now the wave-function of the environment (which as time goes on will include a larger & larger number of degrees of freedom!).

Moreover, one can object that an FAPP distinction is one that can always be overcome with sufficient ingenuity – indeed, recent years have seen experiments demonstrating interference & superposition on an ever larger scale. Thus it is clear that this chapter is not a closed one – and many feel it will only be closed by a superior theory to Q.M.

Let us close with 2 quotes from J.S. Bell (1981):

(a) "Cosmologists must find the usual interpretive rules of QM a bit frustrating. Thus

"... any result of a mmt. of a dynamical variable is one of its eigenvalues..."
"... if the mmt is made a large number of times, the average result will be ..."
"... a mmt causes the system to jump into an eigenstate of the variable being measured..."

It would seem the theory is concerned with "results of mmts", & has nothing else to say. When the system is the whole world, where is the measurer? Inside, presumably. What qualifies some sub-system to play this role? Was the world wave-fn. waiting to jump for billions of yrs, until a single bacterium appeared? Or did it have to wait for an experimenter with a PhD?"

(b) "The concept of "mmt" becomes so fuzzy on reflection that it is surprising to see it appear in physical theory at the most fundamental level. Less surprising is that mathematicians have been able to write extensive works on Q. mmt theory – which experimental physicists do not find it necessary to read."