APPENDIX

SOME PROPERTIES OF FUNCTIONS OF A COMPLEX VARIABLE

Throughout this book we make frequent use of certain properties of functions of a complex variable. A summary of these properties is presented for reference and review in this appendix. We recommend that the student unfamiliar with this material do some outside reading in a mathematics book on the subject. For example, the two little books by Knopp (K4) are quite short and very readable.

A–1 FUNCTIONS OF A COMPLEX VARIABLE. MAPPING

A complex number has the form

$$z = x + iy = re^{i\theta} \tag{A-1}$$

where x, y, r, and θ are real, $i^2 = -1$, and $e^{i\theta} = \cos\theta + i\sin\theta$. x and

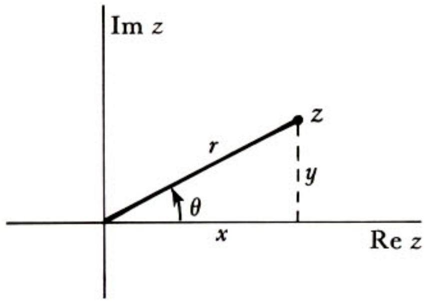

Figure A–1 A point in the complex plane

y are the *real* (Re z) and *imaginary* (Im z) parts of z, respectively, $r = |z|$ is the *magnitude*, and θ is the *phase* or *argument* arg z. Such a number may be represented geometrically by a point on the *complex z-plane*, or *xy*-plane, as shown in Figure A–1. The *complex conjugate* of z will be denoted by z^*; $z^* = x - iy$.

A function $W(z)$ of the complex variable z is itself a complex number whose real and imaginary parts U and V depend on the position of z in the *xy*-plane.

$$W(z) = U(x, y) + iV(x, y) \tag{A-2}$$

Two different graphical representations of the function $W(z)$ are useful. One is simply to plot the real and (or) imaginary parts $U(x, y)$ and $V(x, y)$ as surfaces above the *xy*-plane (see, for example, Section 3–6, Figure 3–13). The other is to represent the complex number $W(z)$ by a point in the complex "*W*-plane," or *UV*-plane, so that to each point in the *z*-plane corresponds one (or more) points in the *W*-plane. In this way, the function $W(z)$ produces a *mapping* of the *xy*-plane onto the *UV*-plane.

EXAMPLE

$$W(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \tag{A-3}$$
$$U = x^2 - y^2 \qquad V = 2xy$$

Alternatively,

$$W = z^2 = r^2 e^{2i\theta}$$

The mapping of a number of points and two curves from the *z*-plane onto the *W*-plane is shown in Figure A–2. For example, the line $x = 1$ becomes the parabola $4U = 4 - V^2$.

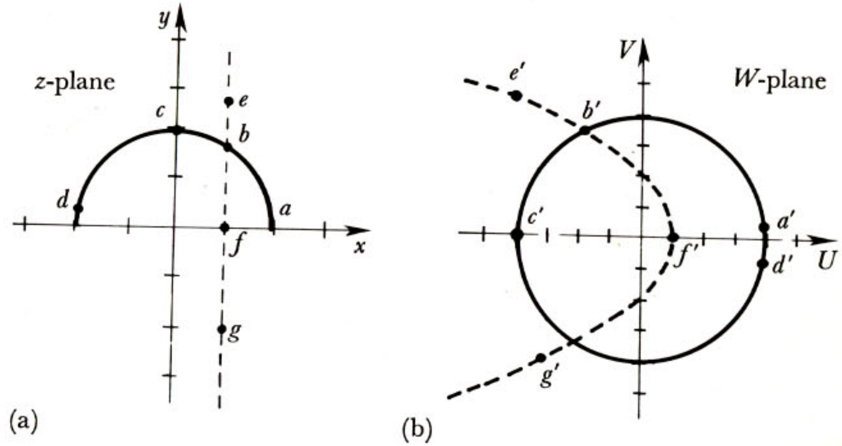

Figure A–2 Illustration by some points and curves of the mapping produced by the function $W(z) = z^2$. The points $a, b, \ldots$ in the z-plane are mapped into $a', b', \ldots$ in the W-plane

In the above example, two points z and $-z$ go into the same point $W(z)$. The upper half of the *z*-plane maps onto the entire *W*-plane, and so does the lower half *z*-plane. Clearly, this situation presents difficulties for the inverse mapping, which is produced by the square root

$$W(z) = z^{1/2} = \sqrt{r}\, e^{i\theta/2} \tag{A-4}$$

This is a *multivalued* function, one point p in the *xy*-plane going into two points p' and p'' in the *UV*-plane. [These are the two square roots corresponding to the phases $\theta_{p'} = \frac{1}{2}\theta_p$ and $\theta_{p''} = \frac{1}{2}(\theta_p + 2\pi)$.] This situation is illustrated in Figure A–3.

Suppose we try to make the mapping single valued by agreeing that a

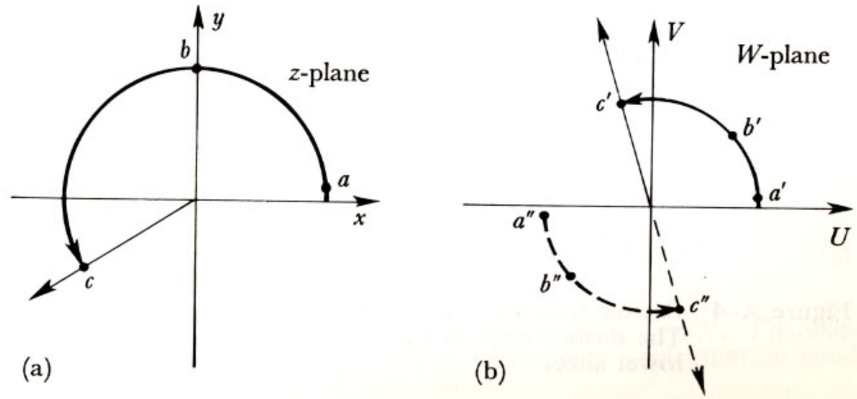

Figure A–3 Illustration of the mapping produced by $W(z) = z^{1/2}$

point p corresponds to p' and not p''. We must make sure that if we start at p and trace a closed curve in the z-plane, the mapping will produce a closed curve in the W-plane starting at p' and returning to p', not p''. This is true provided the closed curve in the xy-plane does not encircle the origin. However, if the curve encircles the origin once, θ changes by 2π and the mapped curve in the W-plane will not return to its starting point.

Thus the multivalued feature can be avoided only if we agree never to encircle the origin $z = 0$. To ensure this, we draw a so-called *branch line* or *branch cut* from $z = 0$ to infinity and agree not to cross it. The singular point $z = 0$ is called a *branch point*. The branch line may be drawn from $z = 0$ to infinity in any way but it is usually convenient to take it along the positive or negative real axis.

The z-plane, when cut in this way, is called a *sheet*, or *Riemann sheet*, of the function $W(z)$. This sheet maps in a single-valued manner onto a portion (in our example, half) of the W-plane, this portion being called a *branch* of the function. A second sheet, similarly cut, is needed to map onto the other half of the W-plane. We may now cross the branch line without getting into multivalue troubles if we transfer from one sheet to the other, when crossing the cut. To picture this, imagine that the edges of the sheets along the cut are joined to each other in the manner indicated in Figure A–4. The sheets so joined form a *Riemann surface* which maps in a single-valued manner onto the entire W-plane. If we now go around the branch point $z = 0$ twice, once

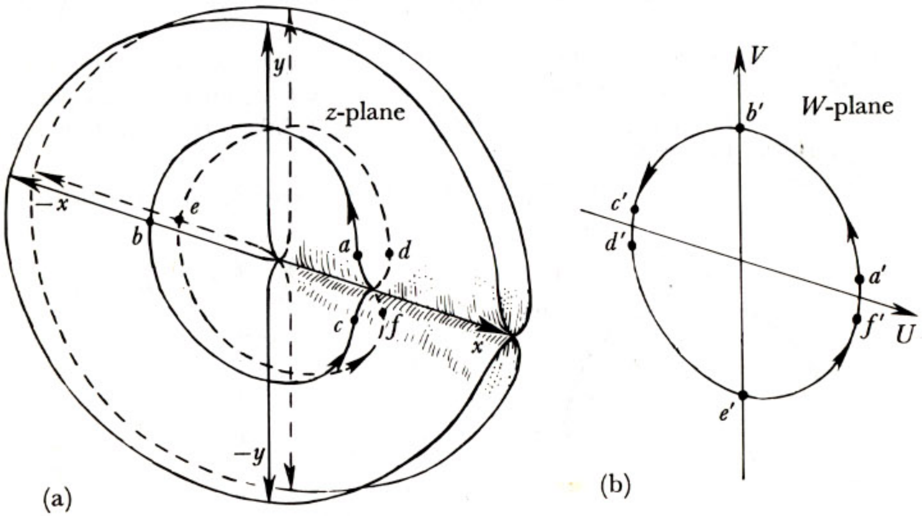

Figure A–4 Riemann surface and mapping for the function $W(z) = z^{1/2}$. The dashed part of the curve in the z-plane lies on the lower sheet

on each sheet, we come back to the starting point in the W-plane, as indicated in Figure A–4.

Other roots may be described in the same way.

EXAMPLE

$$W = z^{1/3} \tag{A-5}$$

The mapping produced by this function is indicated in Figure A–5. The origin is again a branch point, said to be of order 2 because the Riemann surface contains $3(= 2 + 1)$ sheets.

Another example is $W(z) = \ln z$:

$$z = re^{i\theta} \tag{A-6}$$
$$\ln z = \ln r + i\theta$$

The origin is again a branch point, this time of infinite order because the Riemann surface has an infinite number of sheets. Each sheet maps onto a horizontal strip in the W-plane of width $\Delta V = 2\pi$ in the "imaginary" direction. By continued circling of the origin $z = 0$ in the same sense, we never return to the starting point on the map.

Another important type of function is one containing two branch points arising from square roots. Consider

$$W(z) = \sqrt{(z - a)(z - b)}$$

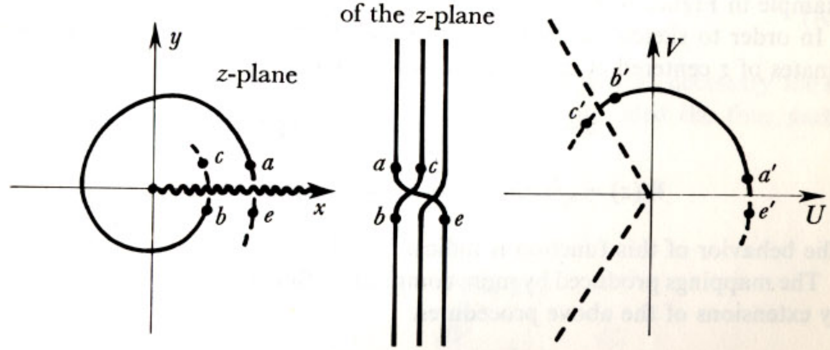

Figure A–5 Riemann surface and mapping for $W = z^{1/3}$. This sketch is a less pictorial way of conveying information similar to that in Figure A–4

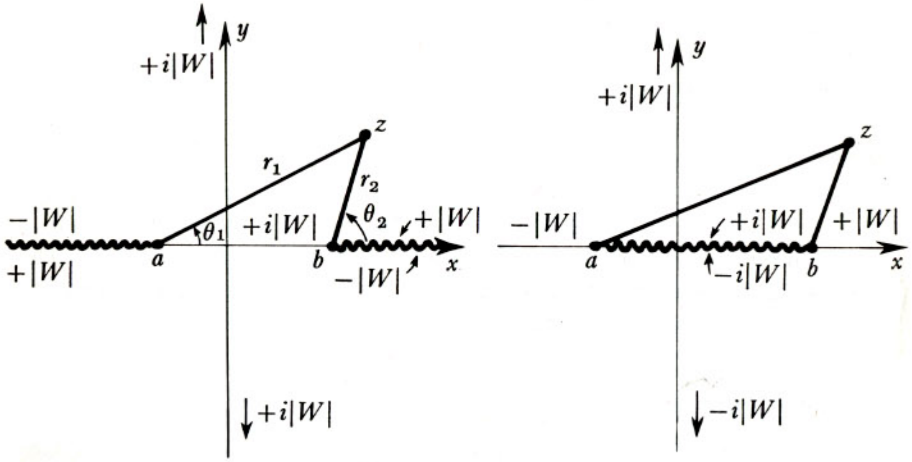

Figure A–6 Two of the ways of drawing branch lines for the function $W(z) = \sqrt{(z-a)(z-b)}$, with the behavior of W indicated in various regions of one Riemann sheet covering the z-plane. In both drawings, the sheet is that one for which $W(z)$ is positive along the "upper side" of the real axis to the right of b. The symbol $-i|W|$, for example, means that $W(z)$ is pure negative imaginary at the place indicated

with branch points at $z = a$ and $z = b$. The Riemann surface of this function may be formed by drawing branch cuts from each of the two branch points to infinity in arbitrary directions, or by making a single cut connecting the two points. The resulting Riemann sheets and the branch of the function corresponding to a given sheet depend on the choice of cuts, as shown for an example in Figure A–6.

In order to sketch the mapping, it is convenient to introduce polar coordinates of z centered at each branch point, that is,

$$z - a = r_1 e^{i\theta_1} \qquad z - b = r_2 e^{i\theta_2}$$

$$W(z) = \sqrt{(z-a)(z-b)} = (r_1 r_2)^{1/2} e^{i\frac{1}{2}(\theta_1 + \theta_2)}$$

The behavior of this function is indicated in Figure A–6.[1]

The mappings produced by more complicated functions may be investigated by extensions of the above procedures.

[1] In the right-hand drawing of Figure A–6, as the point z moves around the cut, the radius vectors from a and b to z will of course sweep across the cut. This is perfectly all right, but the point z itself must not cross the cut—if it does, it will find itself on the other Riemann sheet.

A–2 ANALYTIC FUNCTIONS

In this section we review those properties of analytic functions of a complex variable which are needed in this book. For the general mathematical theory, see any of the numerous books on the subject; for example, Apostol (A5), especially Chapter 16; Knopp (K4); Copson (C8); Whittaker and Watson (W5); or Titchmarsh (T4).

1. A function is *analytic* at a point z if it has a derivative there; that is, if

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{A-7}$$

exists and is independent of the path by which the complex number h approaches zero. If a function is both analytic and single valued throughout a region R, we shall call it *regular* in R. A region of regularity of a multivalued function should be specified on a cut Riemann sheet.

2. If $W(z) = U(x, y) + iV(x, y)$ is an analytic function and we write

$$h = h_x + ih_y$$

then two paths for $h \to 0$ are along the horizontal and vertical directions, for which $h_y = 0$ and $h_x = 0$, respectively. The limits (A-7) obtained for these paths must be equal:

$$\frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{1}{i}\frac{\partial U}{\partial y} + \frac{\partial V}{\partial y} \qquad (= W'(z))$$

Equating real and imaginary parts of this equation gives the *Cauchy–Riemann differential equations*:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \qquad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y} \tag{A-8}$$

These equations may be shown to be sufficient as well as necessary for the function $W = U + iV$ to be regular in a region, provided the four partial derivatives exist and are continuous there.

EXAMPLE

$$W = z^2$$

$$U = x^2 - y^2 \qquad V = 2xy$$

$$\frac{\partial U}{\partial x} = 2x = \frac{\partial V}{\partial y} \qquad \frac{\partial V}{\partial x} = 2y = -\frac{\partial U}{\partial y}$$

The following example shows that some functions are not analytic anywhere.

EXAMPLE

$$W = z^* \qquad \text{(complex conjugate)}$$

$$U = x \qquad V = -y$$

$$\frac{\partial U}{\partial x} = +1 \qquad \frac{\partial V}{\partial y} = -1$$

3. *Integration.* The integral

$$\int_{z_1}^{z_2} f(z)\, dz$$

is a line integral which depends in general on the path followed from z_1 to z_2 (Figure A–7). However, the integral will be the same for two paths if $f(z)$ is regular in the region bounded by the paths. An equivalent statement is *Cauchy's theorem*:

$$\oint_C f(z)\, dz = 0 \tag{A-9}$$

if C is any closed path lying within a region in which $f(z)$ is regular. A kind of converse is also true; if $\oint_C f(z)\, dz = 0$ for every closed path C within a region R, where $f(z)$ is continuous and single valued, then $f(z)$ is regular in R.

4. If $f(z)$ is regular in a region, its derivatives of all orders exist and are regular there.

5. If $f(z)$ is regular in a region R, the value of $f(z)$ at any point within R may be expressed by *Cauchy's integral formula*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{\zeta - z} \tag{A-10}$$

where C is any closed path within R encircling z once in the counterclockwise direction. This formula follows directly from the theorem of residues,

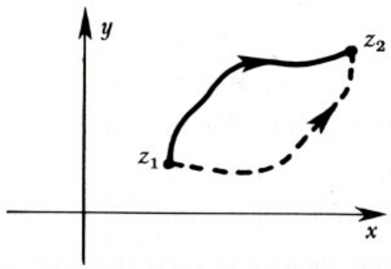

Figure A–7 Paths of integration in the complex plane

item 8 below. The remarkable property of analytic functions implied by Eq. (A-10) should be noted. The values of an analytic function throughout a region are completely determined by the values of the function on the boundary of that region. See Section 5–2 for an application of this property.

Cauchy's formula may be differentiated any number of times to obtain

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)^2}$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n+1}} \tag{A-11}$$

6. A power series expansion (Taylor's series) is possible about any point z_0 within a region where $f(z)$ is regular:

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

$$a_0 = f(z_0) \qquad a_n = \frac{1}{n!} f^{(n)}(z_0) \tag{A-12}$$

The region of the z-plane in which the series converges is a circle. This *circle of convergence* extends to the nearest singularity of $f(z)$, that is, to the nearest point where $f(z)$ is not analytic.

The converse is also true. Any power series convergent within a circle R represents a regular function there.

7. *The Laurent expansion.* If $f(z)$ is regular in an annular region between two concentric circles with center z_0, then $f(z)$ may be represented within this region by a *Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

where the coefficients a_n are

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} \tag{A-13}$$

C is any closed path encircling z_0 counterclockwise within the annular region. Note that the coefficient a_{-1} is

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz \tag{A-14}$$

If $f(z)$ is regular in the annulus, no matter how small we make the inner circle, and yet $f(z)$ is not regular throughout the larger circle, we say that z_0 is an *isolated singularity* of $f(z)$. For such an isolated singularity, there are three possibilities:

(*a*) The Laurent series for $f(z)$ may contain *no* terms with negative powers of $(z - z_0)$. This is a trivial case, and is called a *removable singularity*. By redefining $f(z)$ at the point $z = z_0$, the singularity may be removed. For example, the function

$$f(z) = \begin{cases} z & |z| > 0 \\ 1 & z = 0 \end{cases}$$

has a removable singularity at $z = 0$.

(*b*) The Laurent series for $f(z)$ may contain *a finite number* of terms with negative powers of $(z - z_0)$. In this case z_0 is called a *pole of order m*, where $-m$ is the lowest power of $(z - z_0)$ appearing in the Laurent series. For example, the function $f(z) = (1/\sin z)^2$ has poles of order two at $z = 0, \pm\pi, \pm 2\pi, \ldots$. If $f(z)$ has a pole of order m at z_0, the function $(z - z_0)^m f(z)$ is regular in the neighborhood of z_0.

(*c*) The Laurent series for $f(z)$ may contain *infinitely many* terms with negative powers of $(z - z_0)$. In this case, $f(z)$ is said to have an *essential singularity* at $z = z_0$. For example, $e^{1/z}$ has an essential singularity at $z = 0$ (and therefore e^z has an essential singularity at $z = \infty$).

If z_0 is an isolated singularity, the coefficient a_{-1} in the Laurent expansion is called the *residue* of $f(z)$ at z_0. It has special importance, because of the relation (A-14), as will now be discussed.

8. The *theorem of residues* allows us to evaluate easily the integral of a function $f(z)$ along a closed path C such that $f(z)$ is regular in the region bounded by C except for a finite number of poles and (isolated) essential singularities in the interior of C. By Cauchy's theorem, the path, or *contour*, C may be deformed without crossing any singularities until it is reduced to little circles surrounding each singular point. The integral around each little circle is then given by (A-14), so that we have the theorem of residues

$$\int_C f(z)\, dz = 2\pi i \sum \text{residues} \tag{A-15}$$

where the sum is over all the poles and essential singularities inside C. This theorem is of enormous practical importance in the evaluation of integrals, and a number of examples of its application are given in Section 3–3.

What if a pole lies on the contour? The first thing to do is to look into the physics of the problem to see if this awkward location of the pole results from some approximation. If so, one can decide on which side of the path the pole really lies and thus see whether its residue should be included or not.

A mathematical integral with a pole on the contour strictly does not exist, but, for a simple pole on the real axis, one defines the *Cauchy principal value* as

$$P\int_a^b \frac{f(x)}{x - x_0}\, dx = \lim_{\delta \to 0} \left[\int_a^{x_0 - \delta} \frac{f(x)}{x - x_0}\, dx + \int_{x_0 + \delta}^b \frac{f(x)}{x - x_0}\, dx \right] \tag{A-16}$$

where δ is positive.

The path for the Cauchy principal value integral can form part of a closed contour in which the ends $x_0 \pm \delta$ are joined by a small semicircle centered at the pole (see Figure A–8). Along this semicircle the integral is easy to evaluate; if we let the radius approach zero, $f(z) \to a_{-1}(z - x_0)^{-1}$. Let

$$z - x_0 = re^{i\theta} \qquad dz = ire^{i\theta}\, d\theta$$

Then

$$\int_{\text{semicircle}} f(z)\, dz \to -\int_0^\pi a_{-1} i\, d\theta = -\pi i a_{-1}$$

and if, as is usually the case, the large semicircle gives no contribution,

$$\oint_C f(z)\, dz = P\int f(z)\, dz - \pi i(\text{residue at } z_0)$$
$$= 2\pi i(\textstyle\sum \text{residues inside } C)$$

This gives the result

$$P\int f(z)\, dz = 2\pi i(\tfrac{1}{2} \text{ residue at } x_0 + \textstyle\sum \text{residues inside } C) \tag{A-17}$$

Thus the Cauchy principal value is the average of the two results obtained with the pole inside and outside of the contour.

We often have an integral along the real axis with a simple pole just above (or just below) the axis at x_0. We may consider the pole to be on the axis if we make the path of integration miss the pole by going around x_0 on a little semicircle below (or above). Then it follows by reasoning similar to that leading to (A-17) that the integral may be expressed in terms of the Cauchy principal value as follows:

$$\int \frac{f(x)}{x - x_0 \mp i\varepsilon}\, dx = P\int \frac{f(x)}{x - x_0}\, dx \pm i\pi f(x_0)$$

We may express this result in the somewhat symbolic form

$$\frac{1}{x - x_0 \mp i\varepsilon} = P\frac{1}{x - x_0} \pm i\pi\, \delta(x - x_0) \tag{A-18}$$

where $\delta(x - x_0)$ is the Dirac delta-function defined in (4-19).

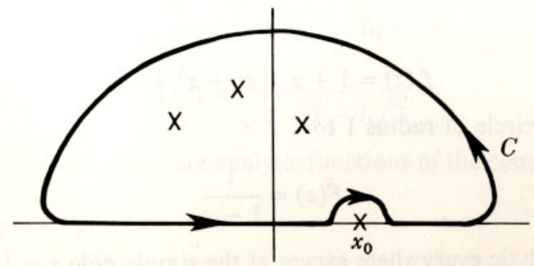

Figure A–8 Illustration of a pole on the real axis

9. The *identity theorem* states that if two functions are each regular in a region R, and have the same values for all points within some subregion or for all points along an arc of some curve within R, or even for a denumerably infinite number of points having a limit point within R, then the two functions are identical everywhere in the region. For example, if $f(z) = 0$ all along some arc in R, then $f(z)$ is the regular function 0 everywhere in R.

This theorem is useful in extending into the complex plane functions defined on the real axis. For example,

$$e^z = 1 + z + \frac{1}{2!}z^2 + \cdots$$

is the unique function $f(z)$ which is equal to e^x on the real axis.

10. Consider a function $f(z)$ which is analytic in a region R of the complex plane, and assume that a finite part of the real axis is included in R. If the function $f(z)$ assumes only *real* values on that part of the real axis in R, then it can be shown that $f(z^*) = [f(z)]^*$ throughout R. That is, going from a point z to its "image" in the real axis, namely, z^*, just carries the *value* f of the function over into *its* image f^*. This is known as the *Schwartz reflection principle*.

The identity theorem forms the basis for the procedure of *analytic continuation*. A power series about z_1 represents a regular function $f_1(z)$ within its circle of convergence, which extends to the nearest singularity. If an expansion of this function is made about a new point z_2, the resulting series will converge in a circle which may extend beyond the circle of convergence of $f_1(z)$. The values of $f_2(z)$ in the extended region are uniquely determined by $f_1(z)$—in fact, by the values of $f_1(z)$ in the common region of convergence of $f_1(z)$ and $f_2(z)$. $f_2(z)$ is said to be the analytic continuation of $f_1(z)$ into the new region. This process may be repeated (with limitations mentioned below) until the entire plane is covered except for singular points by these *elements* of a single function $F(z)$.

EXAMPLE

$$f_1(z) = 1 + z + z^2 + z^3 + \cdots$$

converges in a circle of radius 1 to

$$F(z) = \frac{1}{1 - z}$$

But $F(z)$ is analytic everywhere except at the simple pole $z = 1$, and no other function analytic outside $|z| = 1$ can coincide with $f_1(z)$ within $|z| < 1$. $F(z)$ is the unique analytic continuation of $f_1(z)$ into the entire plane.

Not all functions can be continued indefinitely. The extension may be blocked by a barrier of singularities.

It may also happen that the function $F(z)$ obtained by continuation is multivalued. For example, suppose that after repeating the process described above a number of times, the nth circle of convergence partially overlaps the first one. Then the values of the element $f_n(z)$ in the common region may or may not agree with $f_1(z)$. If they do not agree, then the function $F(z)$ is multivalued, and the "path" along which the continuation was made has encircled one or more branch points.

A power series which converges everywhere defines a single-valued analytic function with no singularities in the entire plane (excluding ∞). Such a function is called an *entire function*. Examples are polynomials, e^z, and $\sin z$. A single-valued function which has no singularities other than poles in the entire plane (excluding ∞) is called a *meromorphic* function. Examples are rational functions, that is, ratios of polynomials.

We conclude by mentioning *Liouville's theorem*; if the function $f(z)$ is regular *everywhere* in the z-plane, including the point at infinity, then $f(z)$ is a constant.

REFERENCES

A very nice treatment of the theory of functions of a complex variable may be found in the two small volumes by Knopp (K4). This subject is treated in many other books, for example, Copson (C8); Whittaker and Watson (W5); Apostol (A5); Nehari (N2); and Titchmarsh (T4).

PROBLEMS

A-1 Describe the mapping produced by the function

$$W(z) = \frac{1}{\sqrt{(z^2 + 1)(z - 2)}}$$

A-2 Describe the mapping produced by the function

$$W(z) = \frac{1}{\sqrt{z - 1 - i\sqrt{2}}}$$

A-3 Which of the following are analytic functions of the complex variable z?

(*a*) $|z|$

(*b*) $\mathrm{Re}\, z$

(*c*) $e^{\sin z}$