SOME BRIEF NOTES ON GREEN FUNCTIONS

The following notes are introductory – it is assumed the reader has only a cursory knowledge of the topic. For more detailed or rigorous discussion, a large number of books are available. Some of those designed for physicists are:

* R. Courant, D. Hilbert

* C.M. Bender, S.A. Orszag, "Advanced Mathematical Methods for Scientists & Engineers", (1978, 1999)

* J. Matthews, R.L. Walker, "Mathematical Methods of Physics" (1964)

* P. Morse, H. Feschbach, "

All of these books discuss Green functions, in one way or another, with one eye on physical applications.

(a) ASSOCIATED EIGENVALUE PROBLEM

: Before beginning with Green functions, let's recall some basic results about eigenvalue problems. These are defined by the equation

$$\hat{L} u(x) = \lambda u(x) \tag{1}$$

where $\hat{L}$ is a linear operator. We will assume that $\hat{L}$ is Hermitian, to make things simple, so that for any pair of functions $u(x)$ and $v(x)$ satisfying the Boundary Conditions (B.C.), we have

$$\int d^D x \; u^*(x) \hat{L} v(x) = \left(\int d^D x \; v^*(x) \hat{L} u(x) \right)^* \tag{2}$$

over the region of interest. (in D dimensions).

We assume a set $\phi_j(x)$ of orthonormal eigenfunctions, so that

$$\int d^D x \; \phi_i(x) \phi_j(x) = \delta_{ij} \tag{3}$$

Then the function $u(x)$ can be expanded uniquely as

$$u(x) = \sum_j u_j \phi_j(x) \tag{4}$$

and completeness implies

$$\sum_j \phi_j^*(x) \phi_j(x') = \delta(x-x') \tag{5}$$

$$u_j = \int d^D x \; \phi_j^*(x) u(x) \tag{6}$$

You are of course used to the Dirac bra/ket representation of these results, in the form

$$
\left.\begin{array}{llll}
(a) & \langle i | j \rangle & = \delta_{ij} & (\text{cf. } (3)) \\
(b) & \langle x | u \rangle & = \sum_j \langle x | j \rangle \langle j | u \rangle \equiv \sum_j u_j \langle x | j \rangle & (\text{cf. } (4)) \\
(c) & \sum_j \langle j | x \rangle \langle x' | j \rangle & = \delta(x - x') & (\text{cf. } (5)). \\
(d) & \langle j | u \rangle & = \langle j | x \rangle \langle x | u \rangle \equiv u_j & (\text{cf. } (6)).
\end{array}\right\} \quad (7)
$$

Notice that for operators $\hat{L}$ that are complicated one may have to slightly modify these definitions. Thus, eg., for the Sturm-Liouville operator, defined by

$$\hat{L}(x) = \left[\frac{d}{dx}\left(p(x) \frac{d}{dx} \right) - q(x) \right] \quad (8)$$

one has the eigenvalue eqtn

$$\hat{L}\, u(x) = \lambda \rho(x)\, u(x) \quad (9)$$

Rather than dividing $\hat{L}$ in (8) by $\rho(x)$ to produce an eigenvalue equation of the form $(L/\rho - \lambda)\, u = 0$ (which would introduce singularities at the zeroes of $\rho(x)$), we can instead modify the orthogonality relations to

$$\langle \phi_i | \phi_j \rangle \equiv \int^{D} dx\, \phi_i^*(x) \rho(x) \phi_j(x) = \delta_{ij} \quad (10)$$

$$\sum_j \langle j | x \rangle \langle x' | j \rangle \equiv \sum_j \phi_j^*(x) \rho(x') \phi_j(x) = \delta(x - x') \quad (11)$$

etc.

(b) <u>DEFINITION of GREEN FUNCTION</u> : Consider now the inhomogeneous linear differential equation, for the same class of linear operators $\hat{L}$:

$$(\hat{L} - \lambda \hat{I})\, \psi(x) = f(x) \quad (12)$$

There are various ways to introduce the Green function for this problem. Here we just do it by defining it as the solution to (12) with $f(x)$ replaced by a delta-function:

$$(\hat{L} - \lambda \hat{I})\, G(x,y) = \delta(x - y) \quad (13)$$

Thus we have a "driving source" for our previous eigenvalue equation, which is just a "point source", or "unit impulse" (these names referring to particular forms of $\hat{L}$).

Using the definition in (13) we can immediately write down a solution to (12). I will do this here assuming open boundaries, to simplify things – we discuss boundary conditions in more detail below.

We then can write that

$$\psi(x) = \int d^D x' \, G(x,x') \, f(x') \qquad (14)$$

To show this is correct, we simply operate on the left of (14) with the operator $\hat{M} = \hat{L} - \lambda \hat{I}$, to get

$$\left.\begin{aligned} \hat{M}\psi(x) &\equiv (\hat{L} - \lambda \hat{I})\,\psi(x) \\ &= (\hat{L} - \lambda \hat{I}) \int dx' \, G(x,x') \, f(x') \\ &= \int dx' \, \delta(x-x') \, f(x') = f(x) \end{aligned}\right\} \qquad (15)$$

Eqtn (14) has a nice intuitive interpretation for any $\hat{L}$. According to (13), $G(x,x')$ is the solution to (12) for a "point source" at x'. Then $\psi(x)$ is the solution for a "sum of point sources", distributed at positions x with magnitude $f(x)$, since

$$f(x) = \int dx' \, f(x') \, \delta(x-x') \qquad (16)$$

Clearly, if we have an explicit form for $G(x,x')$, it can be very useful in finding the solution of a differential equation. So we need to know how to find it.

One simple way is to use an expansion in terms of the eigenfunctions of $\hat{L}$, if these are known. We have

$$\hat{L}\phi_j(x) = \lambda_j \phi_j(x) \qquad (17)$$

Let's expand the solution $\psi(x)$ and also $f(x)$ in terms of the $\phi_j(x)$; we write

$$\psi(x) = \sum_j \psi_j \phi_j(x) \equiv \sum_j \langle j|\psi\rangle \langle x|j\rangle \qquad (18)$$

$$f(x) = \sum_j f_j \phi_j(x) \equiv \sum_j \langle j|f\rangle \langle x|j\rangle \qquad (19)$$

Now from (17) we have that $\hat{L}|j\rangle = \lambda_j |j\rangle$; the inhomogeneous equation (12) then reads $(\lambda_j - \lambda)\psi_j = f_j$, so that

$$\psi_j = \frac{f_j}{\lambda_j - \lambda} \qquad (20)$$

Thus we have

$$\left.\begin{aligned} \psi(x) = \langle x|\psi\rangle &= \sum_j \frac{\langle x|j\rangle\langle j|f\rangle}{\lambda_j - \lambda} \\ &\equiv \sum_j \frac{\langle x|j\rangle\langle j|x'\rangle\langle x'|f\rangle}{\lambda_j - \lambda} \end{aligned}\right\} \qquad (21)$$

But since $\langle x|\psi\rangle = \langle x|G|x'\rangle\langle x'|f\rangle$ from (14), we then have the explicit result that:

$$G(x,x') = \sum_j \frac{\langle x|j\rangle\langle j|x'\rangle}{\lambda_j - \lambda} \equiv \sum_j \frac{\phi_j^*(x')\,\phi_j(x)}{\lambda_j - \lambda} \qquad (22)$$

This eigenfunction expansion for $G(x,x')$ is very useful if we know something about the eigenstates. There are various other ways of finding $G(x,x')$, which we discuss later after dealing explicitly with the role of boundary conditions.

Another important characteristic of $G(x,x')$ follows from the definition in (13). Suppose we specify a form for $\hat{L}$ in terms of the differential operator $d_x \equiv d/dx$. (the generalization to operators involving partial derivatives is straightforward). Let's pick

$$\hat{M}(x) = \sum_{\ell=0}^{n} c_\ell(x) d_x^\ell \equiv \hat{L}(x) - \lambda\hat{I} \qquad (23)$$

Now since $\hat{M}_x G(x,x') = \delta(x-x')$, we have

$$\lim_{\delta\to 0} \sum_{\ell=0}^{n} \int_{x'-\delta}^{x'+\delta} dx\, c_\ell(x)\, d_x^\ell G(x,x') = \lim_{\delta\to 0} \int_{x'-\delta}^{x'+\delta} dx\, \delta(x-x') = 1 \qquad (24)$$

i.e., there is a discontinuity in $\hat{M}G$. This clearly comes from the highest derivative in $\hat{M}$, which is also the highest derivative in $\mathcal{L}$, viz., the term $c_n(x)\, d_x^n$. Thus we have

$$\lim_{\delta\to 0} \int_{x'-\delta}^{x+\delta} dx\, c_n(x)\, d_x^n G(x,x') = 1 \qquad (25)$$

or, after integrating by parts, we have

$$\lim_{\delta\to 0} \left\{ d_x^{n-1} G(x,x') \right\}\Big|_{x'-\delta}^{x+\delta} = 1/c_n(x) \qquad (26)$$

so that the discontinuity in this derivative has magnitude $1/c_n(x)$; lower derivatives of $G(x,x')$ are continuous.

The generalisation of all these results to a space of arbitrary dimensionality is straightforward. The general form for $\hat{M}(\underline{r})$ is

$$\hat{L}(r) - \lambda\hat{I} = \hat{M}(r) = \sum_{\ell=0}^{n} c_{\alpha_1\ldots\alpha_\ell}(\underline{r}) \frac{\partial^\ell}{\partial r_{\alpha_1}\ldots\partial r_{\alpha_\ell}} \qquad (27)$$

where the indices α_j range over the D spatial dimensions. The delta-functions in, eg., eqtn (13), are replaced by D-dimensional delta-fns.; thus, eg., in place of (13) we have

$$\hat{M}(r)\, G(\underline{r},\underline{r}') = \delta(\underline{r}-\underline{r}') \qquad (28)$$