PHYS 501 (Quantum Mechanics): FINAL EXAM

(Tuesday, April 21st, 2009)

12.00 - 2.30 pm, Room Hennings 304

This exam will last 2 hrs and 30 mins. The only material allowed into the exam will be pens, pencils, and erasers. No notes of any kind are permitted, nor any calculators.

There are 2 sections. Students should answer THREE QUESTIONS ONLY from section A, and TWO QUESTIONS ONLY from section B. No extra marks will be given for extra questions answered. The questions in section A should take roughly 15-20 minutes to answer, and the questions in section B roughly 45-50 minutes to answer.

SECTION A

A1: Consider a particle of mass M moving without friction on a circular ring of radius R, centred on the origin in the xy-plane. We insert a flux tube of infinitesimal radius, parallel to $\hat{z}$, through the origin; it carries a total flux Φ. What is the Hamiltonian for the particle in this system? Derive the eigenfunctions and eigenvalues for the particle motion on the ring. Finally, give an expression for the amplitude for the particle to move through an angle θ on the ring in a time t, written as a series expansion over the eigenfunctions.

A2: The Lagrangian for a spin $\mathbf{S}$ is given by $L = S\mathcal{A} \cdot (d\mathbf{n}/dt) - \mathcal{H}$, where $\mathbf{n}(t)$ is a unit vector on the spin sphere, and $\mathcal{H}$ is the Hamiltonian of the spin. The vector $\mathcal{A}$ is the gauge potential from a unit monopole at the centre of the sphere, defined so that $\mathbf{n} \cdot (\nabla \times \mathcal{A}) = 1$.

Show that the equation of motion for $\mathbf{n}(t)$ is given by

$$d\mathbf{n}/dt \;=\; -\mathbf{n} \times \frac{\partial \mathcal{H}}{\partial \mathbf{n}} \tag{1}$$

A3: Consider a Hamiltonian in 3 dimensions given by $\mathcal{H} = \frac{1}{2m}(\mathbf{p} + e\mathbf{A}(\mathbf{r}))^2$, where the vector potential $\mathbf{A}$ describes a magnetic monopole of strength g at the origin, so that

$$\mathbf{B}(\mathbf{r}) \;=\; \frac{g}{4\pi} \frac{\hat{\mathbf{r}}}{r} \tag{2}$$

where $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$.

By considering the flux enclosed by a loop on the surface of a sphere centred on the monopole, show that one can derive Dirac's famous quantization condition $eg = nh$, where n is an integer.

A4: Show, using a path integral argument, a Berry phase argument, or otherwise, that the extra phase accumulated by a particle of charge q going around a ring which encloses a flux Φ is $\phi = 2\pi\Phi/\Phi_o$, where Φ_o is the flux quantum for this system - give an expression for Φ_o.

Now describe in detail an experiment which can be used to verify this 'Aharonov-Bohm' phase, even if the field is zero everywhere on the ring where the particle moves.

A5: Let us write the simple harmonic oscillator Hamiltonian as $\mathcal{H}_o = \frac{1}{2}\hbar\omega[\bar{p}^2 + \bar{x}^2]$, where $\bar{p}$ and $\bar{x}$ are suitably rescaled momentum and position in 1 dimension.

(i) Using appropriate linear combinations of $\bar{p}$ and $\bar{x}$, define ladder operators a, a^+ for the system and show that they satisfy bosonic commutation relations.

(ii) Using the relation $\langle \bar{x}|a|0\rangle = 0$ for the oscillator ground state $|0\rangle$, derive a differential equation for the ground state wave-function $\langle x|0\rangle$, and solve it to find the form of this wave-function.

SECTION B

B1: One may find the propagator for a particle by various methods, and it is useful to combine these. To see this we consider the following problems:

(i) Consider a free particle of mass m, moving in one dimension. Assuming it moves between spacetime points x_1, t_1 and x_2, t_2, find the classical action $\mathcal{S}_c$ for the classical path between these points. The quantum propagator for the same boundary conditions is $G(x_2 - x_1, t_2 - t_1) = A_o e^{i\mathcal{S}_c/\hbar}$, where we need to find A_o. But G is also given by

$$G(x_2 - x_1, t_2 - t_1) = \langle x_2|e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}|x_1\rangle \tag{3}$$

Noting that G is diagonal in the momentum basis, evaluate this expression for G and thereby find the prefactor A_o.

(ii) Now suppose the particle is moving inside a box, ie., it is confined to the region $-L < x < L$ by an infinite potential for $|x| > L$. Find $G(x_2 - x_1, t_2 - t_1)$, as a series expansion over the eigenfunctions of the system.

(iii) Now consider the problem where the particle moves in a linear potential $V(x) = -\alpha x$. What is the classical equation of motion for this system? Find the quantum propagator $G(x_2 - x_1, t_2 - t_1)$ for this system, by first finding the classical action as in (i) above, and then show that the fluctuation prefactor A_o is unchanged from that for the free particle.

B2: We consider a spin-1/2 system with the simple Hamiltonian $\mathcal{H} = \gamma\mathbf{B}(t)\cdot\hat{\boldsymbol{\sigma}}$, where $\hat{\boldsymbol{\sigma}}$ is the Pauli vector, γ is a constant, and $\mathbf{B}(t)$ is a magnetic field with some arbitrary time-dependence.

(i) Writing the wave-function of the spin as $\chi(t) = [a(t)|\uparrow\rangle + b(t)|\downarrow\rangle]$, derive the coupled equations of motion for $a(t)$ and $b(t)$, given the Hamiltonian above. Then, by eliminating $b(t)$ from these equations, derive a single equation of motion for $a(t)$.

(ii) Consider now the special case of the above Hamiltonian where $\mathbf{B}(t)$ is the time-independent constant $\mathbf{B} = \hat{x}\Delta + \hat{z}\epsilon$. Using either the result you have just derived, or by direct calculation, find the amplitude $G_{\uparrow\uparrow}(\tau)$ for the spin to start off at time $t = 0$ in state $|\uparrow\rangle$, and end up at time $t = \tau$ in the same state $|\uparrow\rangle$.

(iii) Again assuming that $\mathbf{B} = \hat{x}\Delta + \hat{z}\epsilon$, find $G_{\uparrow\uparrow}(\tau)$ as a series expansion in $\Delta\tau$, by expanding the expression $G_{\uparrow\uparrow}(\tau) = \langle\uparrow|exp[-i\mathcal{H}\tau/\hbar]|\uparrow\rangle$. Then sum this series in the particular case where $\epsilon_o = 0$.

(iv) Finally, the Hamiltonian of a large spin in a solid can often be written in the above form, again with $\mathbf{B} = \hat{x}\Delta + \hat{z}\epsilon$, where now the two-level system described by $\hat{\boldsymbol{\sigma}}$ simply refers to the two lowest levels of the system. The most interesting case is when $\epsilon_o = 0$; then Δ is just the amplitude for the spin to go from $|\uparrow\rangle$ to $|\downarrow\rangle$. In the simplest cases one can calculate Δ by summing over only 2 paths in a path integral, having amplitudes $A_\pm = \frac{1}{2}\Delta_o e^{\pm i\Phi}$ respectively, where $\Phi = H/H_o + \alpha_o + i\beta_o)$; here H is an applied transverse field, and α_o, β_o are real positive constants. Find the level splitting Δ given by $\Delta^2 = |A_+ + A_-|^2$, and show the result as a function of H in a graph.

B3: One can build up a simple model of measurements using spins. We begin simply:

(i) A system of two spin-1/2 particles is in an initial pure correlated state of form

$$|\Psi\rangle \; = \; a|\uparrow\uparrow\rangle + be^{i\phi}|\downarrow\downarrow\rangle \tag{4}$$

where a, b are real and $a^2 + b^2 = 1$. Write down the density matrix for this system in the 4×4 state basis, showing how you have labelled the states in the matrix. Now, suppose we have no knowledge of the state of the 2nd spin, and must average over it. What is the reduced density matrix for the first spin?

(ii) Now consider the following simple model for a quantum measurement. A particle of mass m_o carrying a single spin $\hat{\boldsymbol{\tau}}_o$ moves in 1 dimension, and interacts with a chain of spins $\hat{\boldsymbol{\sigma}}_j$, with $j = 1, 2, ...N$. The Hamiltonian for the system is

$$\mathcal{H} \; = \; \frac{p_o^2}{2m} + \sum_{j=1}^{N} V(x - x_j)\hat{\sigma}_j^x(1 - \hat{\tau}_o^z) \tag{5}$$

where the particle momentum is p_o, and the interaction $V(x)$ is short-ranged, with $V_o = \int dx V(x) = \pi/4$. The spins $\{\hat{\boldsymbol{\sigma}}_j\}$ are at positions $x_j = ja_o$. Now imagine that the initial state of the system is $|\Psi_{in}\rangle = |\phi_o\rangle|\uparrow\uparrow....\uparrow\rangle$, where the particle wave function is

$$|\phi_o\rangle \; = \; G_{in}(x)e^{ik_o x} \times [a|\Uparrow\rangle + be^{i\phi}|\Downarrow\rangle] \tag{6}$$

and where $G_{in}(x)$ is some very broad incoming envelope function, such that the particle is initially far to the left of the origin (ie., $G_{in}(x) = 0$ for $x > 0$.

Now show that once the particle has completely passed the line of spins, the final state wave-function is

$$|\Psi_f\rangle = G_f(x)e^{ik_o x} \times \left[a|\Uparrow\rangle \prod_j |\uparrow_j\rangle + be^{i\phi}|\Downarrow\rangle \prod_j \downarrow_j\rangle \right] \tag{7}$$

where $G_f(x)$ is the final (even broader) envelope function (and we assume that $G_f(x) = 0$ when $x < na_o$).

(iii) Now you can interpret the preceding result in terms of measurement theory. First consider an initial state $|\Psi\rangle = \sum_j c_j|\phi_j\rangle$ for some quantum system $\mathcal{S}$. Suppose it now interacts with a measuring device $\mathcal{A}$ in such a way that the measuring system final states $|\Phi_j\rangle$ are uniquely correlated with the initial states of the system. Describe how the wave function of the combined system $\mathcal{S} + \mathcal{A}$ evolves during this measurement for either a measurement of the 'first kind' or of the 'second kind'.

Now show how the solution to the model of the spin chain above corresponds to such a measurement scheme. Which kind of measurement does this spin chain model correspond to?

B4: One can learn a lot about scattering theory from simple 1-dimensional scattering. Let us start from the Lippmann-Schwinger integral equation for scattering of an incoming plane wave state $\phi(x) = e^{ikx}$ is

$$\psi^+(x) \;=\; e^{ikx} + \int dx' G_o^+(x - x'; E)\langle x'|V|\psi^+\rangle \qquad \equiv \qquad e^{ikx} + \psi^+_{scatt}(x) \tag{8}$$

where $E = \hbar^2k^2/2m$, and where $G_o(x - x'; E)$ is the 1-particle Green function for a free particle in 1 dimension, so that

$$G_o^+(x - x'; E) \;=\; \langle x|\frac{1}{E - \hat{\mathcal{H}}_o + i\delta}|x'\rangle \tag{9}$$

where $\mathcal{H}_o$ is the free particle Hamiltonian.

(i) Show that the one-particle Green function as defined above is given by

$$G_o^+(x - x'; E) \;=\; -\frac{im}{\hbar^2 k}e^{ik(x-x')} \tag{10}$$

(ii) Now define the T-matrix $T_{kk'} = \langle k'|V|\psi^+\rangle$, and the scattering function $f_{kk'}$ by $\psi^+_{scatt}(x) = f_{kk'}e^{i(kx+\pi/2)}$, where $\psi^+_{scatt}(x)$ is the scattered wave-function defined above in the Lippmann-Schwinger equation. Find $f_{kk'}$ in terms of $T_{kk'}$, and then derive an integral equation for $T_{kk'}$ in terms of the bare scattering potential $V_{kk'} = \langle k|V|k'\rangle$ (starting from the Lippmann-Schwinger equation, or otherwise). Give also the equivalent integral equation for $f_{kk'}$.

(iii) Now suppose the potential $V(x) = V_o\delta(x)$, with $V_o > 0$. Find a geometric series for $T_{kk'}$, and show that this sums to

$$T_{kk'} \;=\; \frac{V_o}{1 + imV_o/\hbar^2 q} \tag{11}$$

where $q = k - k'$. From this construct the scattered wave solution $\psi^+_{scatt}(x)$.

END of EXAM