SOME MOCK EXAM QUESTIONS for PHYS 501 (2009)

REMARKS on the EXAM

The exam will last 2 hrs and 30 mins. The only material allowed into the exam will be pens, pencils, and erasers. No notes of any kind are permitted, nor any calculators.

There will be 2 sections. Students should answer THREE QUESTIONS ONLY from section A, and TWO QUESTIONS ONLY from section B. No extra marks will be given for extra questions answered. The questions in section A should take roughly 15-20 minutes to answer, and the questions in section B roughly 45-50 minutes to answer.

MOCK QUESTIONS

MOCK SECTION A QUESTIONS

A1: Starting from Schrodinger's equation, written as

$$\left[\frac{-\hbar^2\nabla^2}{2m} + V(\mathbf{r})\right]\psi(\mathbf{r},t) \;=\; i\hbar\partial_t\psi(\mathbf{r},t) \tag{1}$$

let us assume a form $\psi(\mathbf{r},t) = A(\mathbf{r},t)e^{i\mathcal{S}(\mathbf{r},t)/\hbar}$ for the wave-function, where $A(\mathbf{r},t)$ is assumed to vary slowly. Find, by expanding in powers of $\hbar$, the equations of motion for $\mathcal{S}(\mathbf{r},t)$ and for the probability $\rho(\mathbf{r}.t) = |\psi(\mathbf{r},t)|^2$. Now, take the equation for $\mathcal{S}$ and, making the identification $E = -\partial_t\mathcal{S}$ for the energy, draw the lines of constant action $\mathcal{S}(\mathbf{r})$ at a given time t for the case where $V(\mathbf{r})$ is a 2-dimensional harmonic oscillator, with $V(\mathbf{r}) = m\Omega_o^2 r^2/2$.

A2: The propagator for a particle in quantum mechanics is the unitary operator producing time evolution of the wave-function, ie.,

$$\hat{G}(t_2 - t_1) = e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)} \tag{2}$$

Show that this operator also satisfies the defining equation for a Green function, viz., that

$$(\hat{H} - i\hbar\partial_t)\hat{G}(t-t') \;=\; -i\hbar\hat{\mathbf{1}}\delta(t-t') \tag{3}$$

where $\hat{\mathbf{1}}$ is the unit operator; and also show, by Fourier transforming the propagator, that it can be written in the position representation as

$$\langle\mathbf{r}|\hat{G}(\omega)|\mathbf{r}'\rangle \;=\; \sum_n \frac{\psi_n^*(\mathbf{r})\psi_n(\mathbf{r}')}{\omega - E_n/\hbar} \tag{4}$$

where the $\{\psi_n\}$ are the eigenfunctions of the Hamiltonian $\hat{H}$.

A3: Consider a free particle of mass m, moving in one dimension. Assuming it moves between spacetime points x_1, t_1 and x_2, t_2, find the classical action $\mathcal{S}_c$ for the classical path between these points. The quantum propagator for the same boundary conditions is $G(x_2 - x_1, t_2 - t_1) = A_o e^{i\mathcal{S}_c/\hbar}$, where we need to find A_o. But G is also given by

$$G(x_2 - x_1, t_2 - t_1) = \langle x_2 | e^{-\frac{i}{\hbar}\hat{H}(t_2 - t_1)} | x_1 \rangle \tag{5}$$

Noting that G is diagonal in the momentum basis, evaluate this expression for G and thereby find the prefactor A_o.

A4: A system of two spin-1/2 particles is in an initial pure correlated stated of form

$$|\Psi\rangle \;=\; a|\uparrow\downarrow\rangle + be^{i\phi}|\downarrow\uparrow\rangle \tag{6}$$

where a, b are real and $a^2 + b^2 = 1$. Write down the density matrix for this system in the 4×4 state basis, showing how you have labelled the states in the matrix. Now, suppose we have no knowledge of the state of the 2nd spin, and must average over it. What is the reduced density matrix for the first spin?

A5: Consider an initial state $|\Psi\rangle = \sum_j c_j |\phi_j\rangle$ for some quantum system $\mathcal{S}$. Suppose it now interacts with a measuring device $\mathcal{A}$ in such a way that the measuring system final states $|\Phi_j\rangle$ are uniquely correlated with the initial states of the system. Explain how the 'von Neumann chain' of measurements works, by describing how the wave function of the combined system $\mathcal{S} + \mathcal{A}$ changes, and how the reduced density matrices of both $\mathcal{S}$ and $\mathcal{A}$ changes, before and after the measurement. You should describe the difference between measurements of the 1st and 2nd kind here.

A6: Consider 2 identical quantum particles which scatter off each other in 3 dimensions through an angle θ, with amplitude $K(\theta)$. Now indistinguishability and unitarity (conservation of probability) mean that we can write $K(\theta) = f(\theta) + e^{2\pi i\alpha} f(\theta + \pi)$. Why is this? Show that in 3 dimensions we must have $\alpha = n/2$.

A7: In 2 dimensions the Hamiltonian for a charged particle in a perpendicular magnetic field is

$$\mathcal{H} \;=\; \frac{1}{2m}(\hat{\mathbf{p}} + e\hat{\mathbf{A}}_o(\mathbf{r}))^2 \;\equiv\; \frac{\hat{\pi}^2}{2m} \tag{7}$$

where we assume that the vector $\nabla \times \mathbf{A}_o(\mathbf{r}) = H_o \hat{z}$. Show first that the commutation relations between the components of π are

$$[\hat{\pi}_x, \hat{\pi}_y] = -i\hbar e H_o \tag{8}$$

Now diagonalise the Hamiltonian by defining bosonic operators which are linear combinations of $\hat{\pi}_x$ and $\hat{\pi}_y$; and thereby find the eigenvalues.

A8: The 1-particle Green function $G_o^{\pm}$ for a free particle with retarded/advanced boundary conditions is

$$G^{\pm}(\mathbf{r}, \mathbf{r}'; E) \;=\; \langle \mathbf{r} | \frac{1}{E - \hat{\mathcal{H}}_o \pm i\delta} | \mathbf{r}' \rangle \tag{9}$$

where the operator $\hat{\mathcal{H}}_o$ has eigenvalues $p^2/2m$. Show, using the diagonal representation of G in momentum space and by contour integration, that in 3 dimensions, the Green function takes the form

$$G^{\pm}(\mathbf{r},\mathbf{r}';E) \;=\; -\frac{m}{2\pi\hbar^2}\frac{e^{\mp ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{10}$$

where k is defined by $E = \hbar^2k^2/2m$.

A9: Find the solutions (eigenvalues and eigenfunctions) to the Schrodinger equation for a 1-d potential in which $V(x) = \infty$ when $|x| > L$, and $V(x) = V_o\delta(x)$ for $|x| < L$.

A10: The partial wave components of the S-matrix are related to the partial wave phase shifts by $S_l = e^{i\delta_l}$, and to the T-matrix components t_l by $S_l = 1 - 2it_l$. Find expressions in terms of the phase shifts for t_l and for $k_l = -\tan\delta_l$ in terms of S_l and in terms of t_l.

Finally, suppose that the $l = 0$ t-matrix component takes the approximate form $t_0(k) = k/(\kappa_o + ik - r_ok^2/2)$, where $\kappa_o = 1/a_0$ is the inverse effective range. Find the form, in this low-momentum expansion, for $\delta_l(k)$ and $S_l(k)$.

A11: The Born approximation of the exact T-matrix is given by $T_{\mathbf{kk}'} = V_{\mathbf{kk}'}$, where $V_{\mathbf{kk}'}$ is the Fourier transform of the real space potential $V(\mathbf{r})$. Show that in 3 dimensions, this is given by

$$V_{\mathbf{kk}'} \;=\; \frac{4\pi}{|\mathbf{k}-\mathbf{k}'|}\int_0^\infty r dr V(r)\sin(|\mathbf{k}-\mathbf{k}'|r) \tag{11}$$

Now suppose that the potential takes the form $V(r) = e^{-\kappa_o r}/r$. Find the form of $V_{\mathbf{kk}'}$.

A12: The Lagrangian for a spin $\mathbf{S}$ is given by $L = S\mathcal{A}\cdot(d\mathbf{n}/dt) - \mathcal{H}$, where $\mathbf{n}(t)$ is a unit vector on the spin sphere, and $\mathcal{H}$ is the Hamiltonian of the spin. The vector $\mathcal{A}$ is the gauge potential from a unit monopole at the centre of the sphere, defined so that $\mathbf{n}\cdot(\nabla\times\mathcal{A}) = 1$.

Show that the equation of motion for $\mathbf{n}(t)$ is given by

$$d\mathbf{n}/dt \;=\; -\mathbf{n}\times\frac{\partial\mathcal{H}}{\partial\mathbf{n}} \tag{12}$$

MOCK SECTION B QUESTIONS