Solution to Assignment 5

1. (a) Notice the following, when $n \neq m$,

$$\langle n \,| m \rangle = 0, \tag{1}$$
$$\langle n | \, H(g) \, | m \rangle = 0. \tag{2}$$

therefore,

$$\nabla_g \langle n \,| m \rangle = 0, \tag{3}$$
$$\nabla_g \langle n | \, H(g) \, | m \rangle = 0. \tag{4}$$

From the last equation, we have

$$\begin{aligned} 0 &= \langle \nabla_g n | \, H(g) \, | m \rangle + \langle n | \, \nabla_g H(g) \, | m \rangle + \langle n | \, H(g) \, | \nabla_g m \rangle \\ &= E_m(g) \langle \nabla_g n \,| m \rangle + \langle n | \, \nabla_g H(g) \, | m \rangle + E_n(g) \langle n | \, \nabla_g m \rangle \\ &= -\left(E_m(g) - E_n(g) \right) \langle n \,| \nabla_g m \rangle + \langle n | \, \nabla_g H(g) \, | m \rangle . \end{aligned} \tag{5}$$

Here in the last step, equ(3) has been used. Therefore,

$$\langle n | \, \nabla_g H(g) \, | m \rangle = \left(E_m(g) - E_n(g) \right) \langle n \,| \nabla_g m \rangle \tag{6}$$

(b) Stokes Theorem in space of $\vec{g}$ gives us,

$$\begin{aligned} \phi_B^n(\mathcal{C}) &= i \oint_{\mathcal{C}} d\vec{g} \cdot \langle n(g) | \, \nabla_g \, n(g) \rangle \\ &= i \iint_{A_{\mathcal{C}}} d\vec{S} \cdot \nabla_g \times \langle n(g) | \, \nabla_g \, n(g) \rangle \\ &= i \iint_{A_{\mathcal{C}}} d\vec{S} \cdot \langle \nabla_g n(g) | \times | \nabla_g n(g) \rangle \\ &= i \iint_{A_{\mathcal{C}}} d\vec{S} \cdot \sum_m \langle \nabla_g n(g) | \, m \rangle \times \langle m \,| \nabla_g n(g) \rangle \\ &= i \iint_{A_{\mathcal{C}}} d\vec{S} \cdot \sum_{m \neq n} \langle \nabla_g n(g) | \, m \rangle \times \langle m \,| \nabla_g n(g) \rangle \end{aligned} \tag{7}$$

In the last step, we used a trick to get rid of n from the summation. It is because that $\langle \nabla_g n \,| n(g) \rangle = -\langle m \,| \nabla_g n(g) \rangle$, so the cross product of them is zero. Order to compare with the formula in the question, one need to furthermore show that the integrand is purely imaginary, then i times that thing is just equal to take its imaginary part and times a -1. Combine equ(6) and equ(7), we have

$$\phi_B^n(\mathcal{C}) = i \iint_{A_{\mathcal{C}}} d\vec{S} \cdot \sum_{m \neq n} \frac{\langle n | \, \nabla_g H(g) \, | m \rangle \times \langle m | \, \nabla_g H(g) \, | n \rangle}{\left(E_m(g) - E_n(g) \right)^2} \tag{8}$$

This equation could be useful in the second question.

2. (a) Initial state is $\frac{1}{2}\left(1+\frac{\sqrt{2}}{2},1,1-\frac{\sqrt{2}}{2}\right)^T$ and Hamiltonian is $H=-2B_0\sigma_z$. So evolution operator is $U = digonal\left(e^{i2B_0t},1,e^{-i2B_0t}\right)$. Therfore final state is $\frac{1}{2}\left(e^{i2B_0t}\left(1+\frac{\sqrt{2}}{2}\right),1,e^{-i2B_0t}\left(1-\frac{\sqrt{2}}{2}\right)\right)^T$. It will finish a circle with period $T=\frac{\pi}{B_0}$. After any number of full circle it comes back to the initial state.

(b) Now let's consider a rotating field, while assuming that the time between the additional field is turned on and off is some integer times the period T, defined accordingly. In this case, the extra phase factor left in the final states is due to Berry phase, not the usual dynamic phase. The new Hamiltonian is $H=-2B_0\hat{g}(t)\cdot\vec{S}$, where $\hat{g}(t)=\frac{\sqrt{2}}{2}(\cos\gamma t,\sin\gamma t,1)^T$ and γ is a small real number. So we have

$$H(\alpha)=\hbar\begin{bmatrix}-\sqrt{2}B_0 & -B_0e^{-i\alpha} & 0\\ -B_0e^{i\alpha} & 0 & -B_0e^{-i\alpha}\\ 0 & -B_0e^{i\alpha} & \sqrt{2}B_0\end{bmatrix} \tag{9}$$

where γt is replaced by α. Now there are several ways to calculate Berry phase. First, find the eigenstates of above Hamiltonian and then plug it into the definition and perform an integral over $\alpha = [0,20\pi]$. Second, make use of equ(8). Let me here try out the first method. *up* state of above Hamiltnoian can be found as

$$|up(\alpha)\rangle=\frac{1}{2}\left(\frac{2+\sqrt{2}}{2}e^{-i\alpha},1,\frac{2-\sqrt{2}}{2}e^{i\alpha}\right)^T. \tag{10}$$

With this at hand, we can use the definition of Berry phase to do the calculation directly.

$$\phi_B^{up}(\mathcal{C})=i\oint_{\mathcal{C}}d\alpha\,\langle up(\alpha)|\,\nabla_\alpha\,up(\alpha)\rangle$$

$$=i\oint_{\mathcal{C}}d\alpha\frac{1}{2}\left(\frac{2+\sqrt{2}}{2}e^{i\alpha},1,\frac{2-\sqrt{2}}{2}e^{-i\alpha}\right)\begin{pmatrix}-i\frac{2+\sqrt{2}}{2}e^{-i\alpha}\\0\\i\frac{2-\sqrt{2}}{2}e^{i\alpha}\end{pmatrix}.$$

$$=i\oint_{\mathcal{C}}d\alpha\frac{1}{4}\left(-i\left(\frac{2+\sqrt{2}}{2}\right)+i\left(\frac{2-\sqrt{2}}{2}\right)\right)$$

$$=\oint_{\mathcal{C}}d\alpha\frac{8\sqrt{2}}{4\times4}$$

$$=10\pi\sqrt{2}. \tag{11}$$

On the other hand, the enclosed solid angle is $\sqrt{2}\pi$, so the total phase is also $10\pi\sqrt{2}$. Of course, any number with a difference of an integer times 2π is the same as above.