Solution to Assignment 4

1. (a) The first part is just a straight forward calculation from $A(E)$ to $T(E)$, and then one uses $T(E)$ to find $G(E)$. Since we are only given $A(E)$ as a function of E, but not the eigenstates, we will not be able to construct the real matrices. So we are good as long as we can figure out $T(E)$ and $G(E)$ as functions of E. Also when construct $G(E)$, in fact $G_0(E)$ is required, but we can assume it is given formally.

$$T(z) = \frac{1}{z+E_0} + \frac{1}{\pi}\mathbb{P}\left[\int_0^\infty \frac{A(\xi)}{\xi - z} d\xi\right]. \tag{1}$$

After using some tricks, say for instance finding and separating all possible poles of the integrand, to deal with principal value integral, one arrives

$$T(z) = \frac{1}{z+E_0} + \frac{N_0\Omega_0^2}{\pi(\Omega_0^2 + z^2)}\left[\ln\frac{\Omega_0}{z} - \frac{z\pi}{2\Omega_0}\right]. \tag{2}$$

Then the full Green's function can be constructed from

$$G(z) = G_0(z) + G_0(z)T(z)G_0(z). \tag{3}$$

(b) The second part is slightly non-trivial. We have calculated exactly the scattering wave for such a potential in our last assignment so exact form of $T(E)$ can be derived easily from there. With Born approximation, the physical picture is totally different. It assumes that the out-going waves have the same form as the in-coming wave. Although it could have possible different frequencies, Born approximation does not take care of the proper coefficients appearing in scattering waves. Because of that, for a *delta* potential scattering wave does have the same form with in-coming wave, however, answers from Born approximation could still be quite different with the exact one. From the last assignment, $T_{-k,k} = \frac{ik\hbar^2}{m}R$ and $T_{k,k} = \frac{ik\hbar^2}{m}S$, where R and S were defined there. For Born approximation,

$$T_{kk'} = \int dx e^{-ik'x} V(x) e^{ik^x} = 2V_0 \cos\frac{(k-k')a_0}{2}. \tag{4}$$

2. (a) The general idea is to find exact form of the radial part of scattering wave $\psi_l(kr)$ and then make use of $f_l(k) = \sqrt{\frac{2}{\pi k}}e^{i\delta_l(k)}\sin\delta_l(k)$ to find $f_l(k)$. And here $\delta_l(k)$ is related to $\psi_l(kr)$ via

$$e^{i\delta_l(k)}\sin\delta_l(k) = -i\frac{m}{2\hbar^2}\int_0^\infty r dr J_l(kr) V(r)\psi_l(kr). \tag{5}$$

As of $\psi_l(kr)$, there are couple ways can be used here: first, solving S.E. by matching boundary conditions as we did before; second, using

L.S.E. and Green's function of free system. Since we are given free Green's function in the notes, here I am going to use the second method. Let's start from LSE in the form of partial waves

$$\psi_l(kr) = J_l(kr) + \int_0^\infty r' dr' V(r') g_0(k, r - r') \psi_l(kr'), \quad (6)$$

where

$$g_0(k, r - r') = -\frac{im}{2\hbar^2} [H_l(kr) J_l(kr') \theta(r - r') + H_l^*(kr') J_l(kr) \theta(r' - r)]. \quad (7)$$

And then plug in the δ potential we will get an equation in terms of $\psi_l(kr)$. In fact due to the property of δ function, this equation is only an equation of $\psi_l(kR_0)$ and a general $\psi_l(kr)$ can be expressed in terms of $J_l(kr)$ and $\psi_l(kR_0)$. Here is the equation of $\psi_l(kR_0)$,

$$\psi_l(kR_0) = J_l(kR_0) - \frac{im}{2\hbar^2} V_0 H_l(kR_0) J_l(kR_0) \psi_l(kR_0). \quad (8)$$

This in turn gives us,

$$\psi_l(kr) = J_l(kr) - \frac{im}{2\hbar^2} \frac{V_0 R_0 J_l(kR_0)}{1 + \frac{im}{2\hbar^2} V_0 H_l(kR_0) J_l(kR_0) \psi_l(kR_0)} \cdot [H_l(kr) J_l(kR_0) \theta(r - R_0) + H_l^*(kR_0) J_l(kr) \theta(R_0 - r)]. \quad (9)$$

And then finally this $\psi_l(kr)$ can be plugged into (5) to get f_l.