Solution to Assignment 3

1. (a) Equation of motion for this 2-d centrally potential can be separated into radial part and angular part, and the radial part becomes,

$$\frac{d^2}{dr^2}R(r)+\frac{1}{r}\frac{d}{dr}R(r)+\left(K^2-\frac{l^2}{r^2}\right)R(r)=0, \tag{1}$$

where l is an integer and $K=\frac{\sqrt{2mE}}{\hbar}$. Solutions of this equation are Bessel functions, generally,

$$R(r)=J_l(Kr)\ \text{ or } Y_l(Kr), \tag{2}$$

and since we require $\lim_{r\to\infty}R(r)=0$, then only J_l is valid. Therefore, we have connection condition at $r=R$ as

$$J_l(KR)=0. \tag{3}$$

This gives us eigenvalues, that only a subset of $\mathbb{R}$ can be our possible values of K and thereafter E. And those are determined by $Z_{n,l}$, the nth root of $J_l(z)$,

$$E=\frac{\hbar^2}{2mR^2}Z_{n,l}^2,\ \text{and}\ ,\psi(r,\theta)\propto e^{il\theta}J_l\left(\frac{Z_{n,l}}{R}r\right). \tag{4}$$

(b) This question can be solved exactly by the same procedure if we regard $V_0\delta(r)$ as $\frac{V_0}{\pi b^2}\theta\left(b^2-r^2\right)$ and then let $b\to 0$ later. Then our general solution is a three-piece function, in respectively three regions $r<b$, $b<r<a$ and $r>a$. And its explicit form depends on the value of E. When $E<0=V(r>a)$, it will be a bounded state while it should be a scattering state when $E>0$. For example, when $E<0$,

$$\begin{cases}\psi(r)=AI_m(\kappa r) & r<b;\\ \psi(r)=BJ_m(Kr)+CY_m(Kr) & b<r<a;\\ \psi(r)=DK_m(kr) & r>a.\end{cases}, \tag{5}$$

where $\kappa=\sqrt{\frac{2m\left(-E+\frac{V_0}{\pi b^2}\right)}{\hbar^2}}$, $K=\sqrt{\frac{2m(E+V_0)}{\hbar^2}}$, and $k=\sqrt{\frac{-2mE}{\hbar^2}}$. Here, one may try to consider bounded states with even lower energy, such as $E<-V_0$. In that case, one could just follow the same procedure and at the end one should find that there is no such bounded state in this question. Next, we need to match all connections.

$$\begin{cases}\psi(b^-)=\psi(b^+), & \psi'(b^-)=\psi'(b^+);\\ \psi(a^-)=\psi(a^+), & \psi'(a^-)=\psi'(a^+).\end{cases} \tag{6}$$

This will relate all coefficients and pick up some specific values of E from $\mathbb{R}$ as it does similarly in part (a). Final results after performing all those algebra and taking the $b \to 0$ limit is

$$\begin{cases} \psi(r) = B_{m,n} J_m(K_{m,n} r) e^{im\theta} & r < a; \\ \psi(r) = D_{m,n} K_m(k_{m,n} r) e^{im\theta} & r > a. \end{cases}, \tag{7}$$

where $K_{m,n}$ and $k_{m,n}$ are determined by eigenenergy $E_{m,n}$, which in turn is determined by the nth root of the following equation,

$$\frac{J_m\left(\sqrt{\frac{2m(E+V_0)}{\hbar^2}}a\right)}{K_m\left(\sqrt{\frac{-2mE}{\hbar^2}}a\right)} = \frac{\sqrt{\frac{2m(E+V_0)}{\hbar^2}} J'_m\left(\sqrt{\frac{2m(E+V_0)}{\hbar^2}}a\right)}{\sqrt{\frac{-2mE}{\hbar^2}} K'_m\left(\sqrt{\frac{-2mE}{\hbar^2}}a\right)} \tag{8}$$

$B_{m,n}$ and $D_{m,n}$ are not independent and can be fixed via normalization. Scattering states can be solved similarly. And this time, the answer is that given an arbitrary value of $E > 0$, we have solution,

$$\begin{cases} \psi(r) = A_m J_m(kr) e^{im\theta} & r < a; \\ \psi(r) = E_m J_m(Kr) e^{im\theta} + F_m Y_m(Kr) e^{im\theta} & r > a. \end{cases}, \tag{9}$$

where $K = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$ and $k = \sqrt{\frac{2mE}{\hbar^2}}$. And this time not all coefficients can be fixed, since normalization condition could not be applied here. But all coefficients are still related.

2. Starting from a plan wave general solution at all three regions and then match their connections at $x = \pm\frac{a}{2}$, although not that trivial because of the δ function, one can easily find the solution,

$$\psi(x) = \begin{cases} e^{ikx} + R_1 e^{-ikx} & r < -\frac{a}{2}; \\ S_1 e^{ikx} + R_2 e^{-ikx} & -\frac{a}{2} < r < \frac{a}{2}; \\ S_2 e^{ikx} & r > \frac{a}{2}. \end{cases}, \tag{10}$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$ and $R_{1,2}, S_{1,2}$ are also functions of k. Another way to solve this question is to make use of the Lippmann-Schwinger equation. It is totally different and it is a very interesting and insightful way to deal with scattering waves. And further more this LSE method can be generalized onto more complicated problems. So try it out if you can.

3. (a) With a flux at only the centre the equation of motion becomes,

$$\left\{\partial_r^2 + \frac{1}{r}\partial_r + \left[k^2 - \frac{1}{r^2}(i\partial_\theta + \alpha)^2\right]\right\}\psi(r,\theta) = 0, \tag{11}$$

which can be separated into angular and radial parts, $\psi(r,\theta) = R(r) e^{il\theta}$, then

$$\left\{\partial_r^2 + \frac{1}{r}\partial_r + \left[k^2 - \frac{1}{r^2}(\alpha - l)^2\right]\right\} R(r) = 0. \tag{12}$$

This is again a Bessel function. Repeat the same procedure in question 1 for the above equation, one can find again bounded states and scattering states for both repulsive and attractive potential. Of course, for repulsive potential, energies of bounded states, if there are such states, have to be $E < 0$. And in this case one can go ahead and do some algebra to check the possibility. In the following I will show an example on bounded states with attractive potential ($V_0 < 0$ and $E - V_0 > 0$).

$$\begin{cases} \psi(r) = A_l J_{l-\alpha}(Kr) e^{il\theta} & r < R; \\ \psi(r) = B_l K_{l-\alpha}(kr) e^{il\theta} & r > R. \end{cases}, \tag{13}$$

where $K = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$ and $k = \sqrt{\frac{-2mE}{\hbar^2}}$. Connection at $r = R$ gives us

$$\begin{cases} A_l J_{l-\alpha}(KR) = B_l K_{l-\alpha}(kR) \\ K A_l J'_{l-\alpha}(KR) = k B_l K'_{l-\alpha}(kR). \end{cases}, \tag{14}$$

This leads to the following equation

$$\frac{J_{l-\alpha}(KR)}{K J'_{l-\alpha}(KR)} = \frac{K_{l-\alpha}(kR)}{k K'_{l-\alpha}(kR)}, \tag{15}$$

which determinds eigenvalues E, and thereafter k, K. And we also notice A_l is related to B_l so in fact there is only one extra degree of freedom which will be fixed by normalization.