SOLUTION to ASSIGNMENT 2

1. (a) The classical path is determined by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \Rightarrow \ddot{x} = -\frac{F(t)}{m}, \tag{1}$$

Therefore, including initial conditions, the final form of $x(t)$ is

$$\begin{aligned} x_c(t) &= -\frac{1}{m}\int_{t_1}^{t} d\tau \int_{t_1}^{\tau} d\tau' F(\tau') \\ &+ \left(\frac{x_2 - x_1}{t_2 - t_1} - \frac{1}{m(t_2 - t_1)}\int_{t_1}^{t_2} d\tau \int_{t_1}^{\tau} d\tau' F(\tau')\right)(t - t_1) + x_1. \end{aligned} \tag{2}$$

Then the classical action can be calculated easily through $S_{cl} = \int_{t_1}^{t_2} dt\left(\frac{m}{2}\dot{x}^2 - F(t)\,x\right) = \frac{m}{2}(x\dot{x})\,|_{t_1}^{t_2} - \frac{1}{2}\int_{t_1}^{t_2} dt F(t)\,x(t)$, and we find

$$\begin{aligned} S_{cl} &= \frac{m(x_2 - x_1)^2}{2(t_2 - t_1)} - \frac{x_2 + x_1}{2}\int_{t_1}^{t_2} d\tau F(\tau) \\ &+ \frac{x_2 - x_1}{2(t_2 - t_1)}\int_{t_1}^{t_2} d\tau \int_{t_1}^{\tau} d\tau' F(\tau') - \frac{x_2 - x_1}{2(t_2 - t_1)}\int_{t_1}^{t_2} d\tau (\tau - t_1) F(\tau) \\ &+ \frac{1}{2m}\int_{t_1}^{t_2} d\tau F(\tau)\int_{t_1}^{\tau} d\tau' \int_{t_1}^{\tau'} d\tau'' F(\tau'') \\ &- \frac{1}{2m(t_2 - t_1)}\int_{t_1}^{t_2} d\tau \int_{t_1}^{\tau} d\tau' F(\tau') \int_{t_1}^{t_2} d\tau'' (\tau'' - t_1) F(\tau''). \end{aligned} \tag{3}$$

(b) To get the propagator, one way is to use the saddle-point approximation, which for the linear potential here is exact, to find the prefactor and classical action separately. Another way is to make use of the known propagator of the SHO, by adding an additional x^2 term. Here we use the first method. The effective Lagrangian for the fluctuation part is

$$L_{eff} = \frac{1}{2}m\dot{q}^2. \tag{4}$$

This is the Lagrangian of a free particle. Using results for the free particle, and combining them with our above classical action, we have

$$G(x_2, t_2; x_1, t_1) = \left(\frac{m}{2\pi i\hbar(t_2 - t_1)}\right)^{\frac{1}{2}} \exp\frac{iS_{cl}}{\hbar}. \tag{5}$$

2. (a) For the 2-level Hamiltonian $H = \epsilon_0 \hat{\tau}_z + \Delta_0 \hat{\tau}_x = \begin{bmatrix} \epsilon_0 & \Delta_0 \\ \Delta_0 & -\epsilon_0 \end{bmatrix}$, the eigenstates are:

$$|+\rangle = \frac{1}{\mathcal{N}_1} \begin{bmatrix} \Delta_0 \\ \sqrt{\epsilon_0^2 + \Delta_0^2} - \epsilon_0 \end{bmatrix}, E_+ = \sqrt{\epsilon_0^2 + \Delta_0^2} \triangleq E \tag{6}$$

$$|-\rangle = \frac{1}{\mathcal{N}_2} \begin{bmatrix} -\Delta_0 \\ \sqrt{\epsilon_0^2 + \Delta_0^2} + \epsilon_0 \end{bmatrix}, E_- = -\sqrt{\epsilon_0^2 + \Delta_0^2} = -E, \tag{7}$$

where

$$\begin{cases} \mathcal{N}_1 = \sqrt{\left(\sqrt{\epsilon_0^2 + \Delta_0^2} - \epsilon_0\right)^2 + \Delta_0^2} \\ \mathcal{N}_2 = \sqrt{\left(\sqrt{\epsilon_0^2 + \Delta_0^2} + \epsilon_0\right)^2 + \Delta_0^2} \end{cases} \tag{8}$$

In order to simplify the notation, let us assume that all parameters are real, and define $\cos\alpha = \frac{\Delta_0}{\mathcal{N}_1}, \sin\alpha = \frac{\sqrt{\epsilon_0^2+\Delta_0^2}-\epsilon_0}{\mathcal{N}_1}$; then

$$|+\rangle = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, E_+ = E \tag{9}$$

$$|-\rangle = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}, E_- = -E, \tag{10}$$

To find the time evolution of the density matrix corresponding to a pure state $|\psi(0)\rangle$, one could first calculate $|\psi(t)\rangle$ and then use it to construct the density matrix via $\rho(t) = |\psi(t)\rangle \langle\psi(t)|$. Alternatively one can first determine the evolution operator $U(t) = e^{-iHt}$, and apply this to the initial density matrix via $\rho(t) = U(t)\rho(0)U^\dagger(t)$. Here we use the latter method. The matrix elements of $U(t)$ can be calculated as follows. The first one is given by

$$\begin{aligned} U_{11} = \langle\uparrow| e^{-iHt} |\uparrow\rangle = & \langle\uparrow| e^{-iHt} |+\rangle \langle + |\uparrow\rangle + \langle\uparrow| e^{-iHt} |-\rangle \langle - |\uparrow\rangle \\ = & e^{-iE_+t} \langle\uparrow|+\rangle \langle + |\uparrow\rangle + e^{-iE_-t} \langle\uparrow|-\rangle \langle - |\uparrow\rangle \\ = & e^{-iEt}\cos^2\alpha + e^{iEt}\sin^2\alpha \\ = & \cos Et - i\sin Et\cos 2\alpha. \end{aligned} \tag{11}$$

and the other elements are found in the same way. But since here the initial state is exactly $|\uparrow\rangle$, we only need one more element,

$$\begin{aligned} U_{21} = \langle\downarrow| e^{-iHt} |\uparrow\rangle = & e^{-iEt}\cos\alpha\sin\alpha - e^{iEt}\cos\alpha\sin\alpha \\ = & -i\sin Et\sin 2\alpha. \end{aligned} \tag{12}$$

Thence we have

$$\rho(t) = U(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U^{\dagger}(t) = \begin{bmatrix} U_{11}U_{11}^{*} & U_{11}U_{21}^{*} \\ U_{21}U_{11}^{*} & U_{21}U_{21}^{*} \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 Et + \frac{\epsilon_0^2}{E^2}\sin^2 Et & \frac{\epsilon_0\Delta_0}{E^2}\sin^2 Et - i\frac{\Delta_0}{E}\sin Et\cos Et \\ \frac{\epsilon_0\Delta_0}{E^2}\sin^2 Et + i\frac{\Delta_0}{E}\sin Et\cos Et & \frac{\Delta_0^2}{E^2\sin^2 Et} \end{bmatrix} \tag{13}$$

(b) On the other hand, using the expansion $e^{i\theta\sigma_r} = \cos\theta + i\sin\theta\sigma_r$, the propagators can be written down directly. Thus

$$\begin{aligned} G &= e^{-i(\epsilon_0\hat{\tau}_z + \Delta_0\hat{\tau}_x)t} \\ &= e^{-iE\sigma_r t} = \cos Et + \sigma_r \sin Et, \end{aligned} \tag{14}$$

where $E = \sqrt{\Delta_0^2 + \epsilon_0^2}$ and $\sigma_r = \begin{bmatrix} \frac{\epsilon_0}{E} & \frac{\Delta_0}{E} \\ \frac{\Delta_0}{E} & -\frac{\epsilon_0}{E} \end{bmatrix}$. The elements of this 2×2 matrix are easily calculated; thus, eg., $G^{\uparrow\uparrow} = \cos Et - i\frac{\epsilon_0}{E}\sin Et$. With this propagator, the top left element of the density matrix is found to be

$$\begin{aligned} \rho_{\uparrow\uparrow} &= \langle\uparrow\left|e^{-iHt}\right|\uparrow\rangle\langle\uparrow|e^{iHt}\left|\uparrow\right\rangle = G^{\uparrow\uparrow}\left(G^{\uparrow\uparrow}\right)^{*} \\ &= \cos^2 Et + \frac{\epsilon_0^2}{E^2}\sin^2 Et. \end{aligned} \tag{15}$$

which agrees with equ(13).

3. Our Hamiltonian is
$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{m}{2}\left(x_1^2 + x_2^2 + 2gx_1x_2\right)$. In general one can apply the linear matrix transformation $\begin{bmatrix} 1 & g \\ g & 1 \end{bmatrix}$ to the original coordinates to decouple x_1, x_2 in the original Hamiltonian. In this case we have

$$y_1 = \frac{x_1 + x_2}{\sqrt{2}}, \qquad y_2 = \frac{x_1 - x_2}{\sqrt{2}}. \tag{16}$$

In terms of these new variables,

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{m}{2}(1+g)y_1^2 + \frac{m}{2}(1-g)y_2^2. \tag{17}$$

This is a system of two decoupled SHOs, with the form

$$H = \hbar\omega_1\left(a_1^{\dagger}a_1 + \frac{1}{2}\right) + \hbar\omega_2\left(a_2^{\dagger}a_2 + \frac{1}{2}\right), \tag{18}$$

in terms of creation and annihilation operators defined as

$$a_1 = \frac{1}{\sqrt{2m\hbar\omega_1}}\left(ip_1 + m\omega_1 y_1\right), \quad a_2 = \frac{1}{\sqrt{2m\hbar\omega_2}}\left(ip_2 + m\omega_2 y_2\right), \tag{19}$$

where $\omega_{1,2} = \sqrt{1 \pm g}$.

(b) A coherent state for this system can be written as

$$|z_1, z_2\rangle = \frac{1}{\pi} e^{-\frac{z_1^* z_1 + z_2^* z_2}{2}} \sum_{n_1, n_2} \frac{z_1^{n_1}}{\sqrt{n_1!}} \frac{z_2^{n_2}}{\sqrt{n_2!}} |n_1, n_2\rangle . \tag{20}$$

Now let us consider $\langle x_1(t)\rangle, \langle p_1(t)\rangle, \langle x_2(t)\rangle, \langle p_2(t)\rangle$ and if necessary also $\langle \delta x_1(t)\rangle, \langle \delta p_1(t)\rangle, \langle \delta x_2(t)\rangle, \langle \delta p_2(t)\rangle$. Here we just find the first set of these:

$$\langle x_1(t)\rangle = \sqrt{\frac{\hbar}{2m\omega_1}} \langle z_1, z_2| a_1^\dagger(t) + a_1(t) |z_1, z_2\rangle \tag{21}$$

$$= \sqrt{\frac{\hbar}{2m\omega_1}} \langle z_1(t), z_2(t)| a_1^\dagger + a_1 |z_1(t), z_2(t)\rangle . \tag{22}$$

From either of these expression we can extract an intuitive picture of the motion. For example, from the first one, we find that $\langle x_1(t)\rangle \propto Re\left(z_1 e^{-i\omega_1 t}\right)$. We likewise find that $\langle p_1(t)\rangle \propto Im\left(z_1 e^{-i\omega_1 t}\right)$. Thence the complex variable $z_1 = \langle x_1(t)\rangle + \langle p_1(t)\rangle$ rotates periodically in the xp plane.