Phys 501: HOMEWORK ASSIGNMENT No (3)

Wednesday Feb 24th 2010

DUE DATE: Friday March 12th 2010.
Assignments handed in late may not receive a full mark.

QUESTION (1): QUANTUM MEASUREMENTS - A TOY MODEL

(i) A system of two spin-1/2 particles is in an initial pure correlated state of form

$$|\Psi\rangle \;=\; a|\uparrow\uparrow\rangle + be^{i\phi}|\downarrow\downarrow\rangle \tag{1}$$

where a, b are real and $a^2 + b^2 = 1$. Write down the density matrix for this system in the 4×4 state basis, showing how you have labelled the states in the matrix. Now, suppose we have no knowledge of the state of the 2nd spin, and must average over it. What is the reduced density matrix for the first spin?

(ii) Now consider the following simple model for a quantum measurement. A particle of mass m_o carrying a single spin $\hat{\boldsymbol{\tau}}_o$ moves in 1 dimension, and interacts with a chain of spins $\hat{\boldsymbol{\sigma}}_j$, with $j = 1, 2, ...N$. The Hamiltonian for the system is

$$\mathcal{H} \;=\; \frac{p_o^2}{2m} + \sum_{j=1}^{N} V(x - x_j)\hat{\sigma}_j^x(1 - \hat{\tau}_o^z) \tag{2}$$

where the particle momentum is p_o, and the interaction $V(x)$ is short-ranged, with $V_o = \int dx V(x) = \pi/4$. The spins $\{\hat{\boldsymbol{\sigma}}_j\}$ are at positions $x_j = ja_o$. Now imagine that the initial state of the system is $\Psi_{in}(x,t)\rangle = \psi_o(x, \hat{\boldsymbol{\tau}}_o; t)\rangle| \uparrow\uparrow .... \uparrow\rangle$, where the particle wave function is

$$|\psi_o\rangle \;=\; G_{in}(x,t) \times [a|\Uparrow\rangle + be^{i\phi}|\Downarrow\rangle] \tag{3}$$

and where $G_{in}(x,t)$ is some very broad incoming envelope function, such that the particle is initially far to the left of the origin; we will choose a function which satisfies the free particle Schrodinger equation, of form:

$$G_{in}(x,t) \;=\; W(t)e^{-\pi(x-vt)^2/W^2(t)}e^{ik(x-vt)} \tag{4}$$

where $W(t = 0) = W_o \gg a_o$ when $t = 0$.

Now show that once the particle has completely passed the line of spins, the final state wave-function is

$$|\Psi_f\rangle = G_f(x,t) \times \left[a|\Uparrow\rangle \prod_j |\uparrow_j\rangle + be^{i\phi}|\Downarrow\rangle \prod_j |\downarrow_j\rangle\right] \tag{5}$$

where $G_f(x,t)$ is the final (even broader) envelope function (and we assume that $G_f(x) \sim 0$ when $x < na_o$, ie., it has left the vicinity of the spin chain). Evaluate $G_f(x,t)$ under these conditions. *NB: Ignore any back-scattering of the particle off the potential.*

(iii) Now interpret the preceding result in terms of measurement theory. First consider an initial state $|\Psi\rangle = \sum_j c_j|\phi_j\rangle$ for some quantum system $\mathcal{S}$. Suppose it now interacts with a measuring device $\mathcal{A}$ in such a way that the measuring system final states $|\Phi_j\rangle$ are uniquely correlated with the initial states of the system. Describe how the wave function of the combined system $\mathcal{S} + \mathcal{A}$ evolves during this measurement for either a measurement of the 'first kind' or of the 'second kind'.

Now show how the solution to the model of the spin chain above corresponds to such a measurement scheme. Find the reduced density matrices of (a) the particle once it has left the spin chain, and (b) of the spin chain itself.

Which kind of measurement does this spin chain toy model correspond to?

QUESTION (2): 1-D TWO-WELL PROBLEM

Consider a 1-dimensional "double-barrier" potential of form

$$V(x) \;=\; -V_o[\delta(x - a_o/2) + \delta(x + a_o/2)] \tag{6}$$

where each δ-function potential well has the same strength.

Find the solutions (eigenfunctions and eigenenergies) to this problem assuming we have a plane wave coming in from the left.

QUESTION (3): 2-D REPULSIVE POTENTIAL with ADDED FLUX TUBE

Consider the problem of a charged particle moving in a problem which has rotational symmetry about the origin. The particle moves simultaneously in a 2-d 'circular' potential with the form

$$V(r) \;=\; 0 \quad (r > R); \qquad\qquad V(r) \;=\; V_o \quad (r < R) \tag{7}$$

which we assume to be a repulsive, with $V_o > 0$, and with $R > 0$; and in the field of a flux tube of infinitesimal radius, but containing a total flux $\Phi = \alpha\Phi_o$, where $\Phi_o = h/e$ is the flux quantum and e the charge of the particle (if the radius of the flux tube is $a_o < R$, then the field has strength $B(r) = (\Phi/\pi a_o^2)\theta(a_o - r)$; then let $a_o \rightarrow 0$).

(i) Write down the Hamiltonian for this problem, assuming a finite value for a_o.

(ii) Now solve the problem for the particle moving simultaneously in the potential and the flux tube field just specified, now assuming $a_o \rightarrow 0$, finding the eigenfunctions and eigenvalues. *NB: Do NOT try to solve this problem inside the flux tube - this is more complicated and requires a solution in terms of hypergeometric functions. You can simply assume that the flux tube is infinitesimal in radius.*

(iii) Now find the one particle propagator $G(r_2\theta_2; r_1\theta_1; t)$ for this system, as an expansion over eigenfunctions. Finally - what does this result reduce to in the limit where $V_o \rightarrow \infty$, and $R \rightarrow 0$ (still assuming that $R > a_o$)?