Phys 501: HOMEWORK ASSIGNMENT No (1)

Tuesday January 31st 2012

DUE DATE: Friday Feb 17th 2011.

(Please note that assignments handed in late may not receive a full mark.)

QUESTION (1): DENSITY MATRICES and MEASUREMENTS

(i) Consider a 2-spin system of spin-1/2 particles, which has been prepared such that we have (a) a probability 1/2 that the system is in an incoherent mixture, with equal weighting, of the 3 triplet states $|\psi_0^T\rangle = [|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]/\sqrt{2}$, $|\psi_+^T\rangle = |\uparrow\uparrow\rangle$, and $|\psi_-^T\rangle = |\downarrow\downarrow\rangle$; and (b) a probability 1/2 that the system is in a pure state with singlet wave-function $|\psi_S\rangle = [|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]/\sqrt{2}$.

Find the total density matrix for this 2-spin system, quantizing along the $\hat{z}$-axis. Then calculate the expectation values of (a) the "up-up" operator $\hat{\sigma}_1^z\hat{\sigma}_2^z$, and (b) the "up-right" operator $\hat{\sigma}_1^z\hat{\sigma}_2^x$, for the pair of spins.

(ii) Now suppose we trace over spin 2, to get a reduced density matrix for spin 1. Find this reduced density matrix, and then calculate the expectation values of $\hat{\sigma}_1^z$, and $\hat{\sigma}_1^x$, for the first spin.

QUESTION (2): PROPAGATORS

The propagator for a particle moving in 1 dimension in quantum mechanics can be written in the position/energy representation as

$$G(x, x'; \omega) \;=\; \langle x|\hat{G}(\omega)|x'\rangle \;=\; i\hbar \sum_n \frac{\phi_n^*(x)\phi_n(x')}{\hbar\omega - \epsilon_n} \tag{1}$$

where the $\langle x|n\rangle = \phi_n(x)$ are the eigenfunctions of the Hamiltonian, and ϵ_n are the eigenenergies.

(i) Find an expression for $G(x, x'; \omega)$ for a free particle of mass m.

(ii) Now find $G(x, x'; \omega)$ for a particle of mass m, confined to a square well potential with infinitely high sides, and a width $2L$ (ie., with $V(x) = 0$ for $|x| < L$, and $V(x) = \infty$ for $|x| > L$).

(iii) Now consider a harmonic oscillator, of mass m and frequency ω_o. Let us suppose we know that the propagator for a simple harmonic oscillator is

$$G(x_1, x_2; t) \;=\; \left(\frac{-im\omega_o}{2\pi\hbar\sin\omega_o t}\right)^{1/2} \exp\left(\frac{im\omega_o}{2\hbar\sin\omega_o t}[(x_1^2 + x_2^2)\cos\omega_o t - 2x_1x_2]\right) \tag{2}$$

but that we do not know the eigenfunctions or eigenvalues of the Hamiltonian. One can in principle find complete expressions for these, by comparing $G(x_1, x_2; t)$ in the form (2) with the eigenfunction expansion $G(x_1, x_2; t) = \sum_n \phi_n^*(x_1)\phi_n(x_2)\exp[(-i/\hbar)\epsilon_n t]$, derived by Fourier transforming (1) (this will work if you have an intimate understanding of Hermite polynomials). However in what follows, show by expanding the sine and cosine functions in (2) in power series, and comparing with the terms in the eigenfunction expansion, that the eigenvalues of the oscillator are given by $\epsilon_n = \hbar\omega_o(n+1/2)$. Then, by picking out the coefficient of the lowest term, show that the ground state eigenfunction is

$$\phi_o(x) \;=\; \left(\frac{m\omega_o}{\pi\hbar}\right)^{1/4} \exp[-m\omega_o x^2/2\hbar] \tag{3}$$

QUESTION (3): PATH INTEGRALS

(i) Consider a free particle moving between spacetime points x_1, t_1 and x_2, t_2. Find the action $\mathcal{S}_o$ for the classical path between these points. The quantum propagator between the points is $G(x_2 - x_1, t_2 - t_1) = A_o e^{i\mathcal{S}_o/\hbar}$, with prefactor A_o. However G is also given by

$$G_o(x_2 - x_1, t_2 - t_1) = \langle x_2 | e^{-\frac{i}{\hbar}\hat{H}_o(t_2 - t_1)} | x_1 \rangle \tag{4}$$

Noting that G is diagonal in the momentum basis, evaluate this expression for G and thereby find the prefactor A_o.

(ii) Now let us consider the following modification to quantum mechanics. Suppose we have the following expression for the propagator between 2 spacetime points:

$$\mathcal{G}(x_2, x_1; t_2, t_1) \;=\; G_o(x_2, x_1; t_2, t_1) + \Delta\mathcal{G}(x_2, x_1; t_2, t_1) \tag{5}$$

where $G_o(x_2, x_1; t_2, t_1)$ is given by the usual path integral expression, viz.,

$$G_o(x_2, x_1; t_2, t_1) \;=\; \int_{x_1}^{x_2} \mathcal{D}q(\tau) \exp \frac{i}{\hbar} \int_{t_1}^{t_2} d\tau L(q, \dot{q}; \tau) \tag{6}$$

and the correction is given by

$$\Delta\mathcal{G}(x_2, x_1; t_2, t_1) \;=\; \int_{x_1}^{x_2} \mathcal{D}q_1(\tau) \int_{x_1}^{x_2} \mathcal{D}q_2(\tau) K(q_1, q_2) \exp \frac{i}{2\hbar} \int_{t_1}^{t_2} d\tau \;\; [L(q_1, \dot{q}_1; \tau) \;+\; L(q_2, \dot{q}_2; \tau)] \tag{7}$$

We would like to explore this idea by looking at some simple cases. Let us begin by assuming that we are dealing with a free particle (no external forces), so that G_o is given by the result in (i) above. Consider first the case where $K(q_1, q_2) = \lambda\delta(q_1 - q_2)$. Find $\Delta\mathcal{G}(x_2, x_1; t_2, t_1)$ for this case, and comment on the physical significance of the result.

(iii) Now let us assume that $K(q_1, q_2) = \lambda \exp[-(q_1 - q_2)^2/2\sigma]$, where σ is some range parameter for the correlation function $K(q_1, q_2)$. One way to go about dealing with this is to assume that $\lambda \ll 1$, so that we are dealing with a small perturbation. In this case, you can try to approximate the answer by assuming that the wave-function is focussed along the same classical path or paths as it would be in ordinary quantum mechanics, ie., that the exponents in the path integrations are unchanged by the small additional term). Using this approximation, or otherwise, find out what is $\mathcal{G}(x_2, x_1; t_2, t_1)$ for this case. Bonus marks for those who do not use the approximation that $\lambda \ll 1$.

Finally, comment on the physical significance of the answer - in what way could you investigate the difference between results predicted by this form for $\mathcal{G}(x_2, x_1; t_2, t_1)$, and by standard quantum mechanics?