Phys 501: HOMEWORK ASSIGNMENT No (4)

Tuesday March 29th 2011

DUE DATE: Friday April 8th 2011.

(Please note that assignments handed in late may not receive a full mark.)

QUESTION (1): ADIABATIC REGIME

(i) Consider the time-dependent Schrodinger equation $\mathcal{H}(t)\Psi(t) = i\hbar\partial_t\Psi(t)$ in the limit where $\mathcal{H}(t)$ changes slowly as a function of some parameter $\mathbf{g}(t)$. Defining the adiabatic eigenstates $\psi_n(t)$ by $\mathcal{H}(t)\psi_n(t) = E_n(t)\psi_n(t)$, where now t is simply a fixed parameter, write $\Psi(t)$ in the form

$$\Psi(t) \; = \; \sum_n c_n(t)\psi_n(t)e^{\frac{-i}{\hbar}\int^t dt' E_n(t')} \tag{1}$$

and show that the equation of motion for the $c_n(t)$ can be written as

$$\dot{c}_n(t) \; = \; -\sum \langle\psi_n|\partial_t|\psi_m\rangle c_m(t)\; e^{\frac{i}{\hbar}\int^t dt'(E_n(t')-E_m(t'))} \tag{2}$$

Show also that

$$\langle\psi_n|\partial_t|\psi_m\rangle \; = \; -\frac{\langle\psi_n|\partial_t\mathcal{H}|\psi_m\rangle}{(E_n(t)-E_m(t))} \tag{3}$$

when $m \neq n$.

(ii) Now consider the case where $m = n$. We imagine taking the Hamiltonian slowly around a circuit, by varying $\mathbf{g}(t)$ over a long time period so as to bring it back to its original value. If the Berry phase is defined as $\phi_B^n(\mathcal{C}) = i\oint_{\mathcal{C}} d\mathbf{g}\cdot\langle n(g)|\nabla_g n(g)\rangle$, where the line integral is taken around the circuit $\mathcal{C}$, then show that

$$\begin{aligned}\phi_B^n(\mathcal{C}) \; &= \; -Im\oint_{\mathcal{C}} d\mathbf{S}\cdot\nabla_g\times\langle n|\nabla_g n\rangle \\ &= \; -Im\oint_{\mathcal{C}} d\mathbf{S}\cdot\sum_{m\neq n}\langle\nabla_g n|m\rangle\times\langle m|\nabla_g n\rangle\end{aligned} \tag{4}$$

where the integration is over the surface in parameter space (ie., $\mathbf{g}$-space) enclosed by the closed curve $\mathcal{C}$.

QUESTION (2): SPIN DYNAMICS

(i) Suppose we have a Hamiltonian $\mathcal{H}_o(\hat{\boldsymbol{\sigma}})$ for a spin-half system taking the simple form

$$\mathcal{H}_o(t) \; = \; B_z\hat{\tau}_z + b_x\hat{\tau}_x\theta(t) \tag{5}$$

Suppose the system starts for $t < 0$ in the state $|\uparrow\rangle$; find the solution to the Schrodinger equation for this system for $t > 0$, and show it is precessional motion. What is the solid angle subtended by one circuit of this precessional motion?

(ii) A spin $S = 1/2$ is oriented at time $t = 0$ in the xz-plane along an angle 60^o between the x and z axes. A magnetic field B_o along the z-axis is applied at $t = 0$. Assuming an electronic

g-factor 2, determine the precession period T for the spin. Then, suppose we allow the spin to precess for a time $t = 100T$. What is the final state of the spin wave-function?

(iii) Now suppose we have the same spin in the same initial state as in (i), but we now apply a field of strength B_o at $t = 0$ which is parallel to the spin's initial state. We then move this field very slowly in a circuit around the z-axis, completing 100 full circuits, always keeping the field at an angle of 60^o from the z-axis. What now is the final state of the spin wave-function?