Phys 501: HOMEWORK ASSIGNMENT No (3)

Monday March 6th 2011

DUE DATE: Wed March 23rd 2011.

(Please note that assignments handed in late may not receive a full mark.)

QUESTION (1): PERTURBATION THEORY for SPIN

Suppose we have a spin of magnitude S in a combination of easy-axis and 4th-order anisotropy fields, where the Hamiltonian is given by

$$\mathcal{H}_o = -K_o \hat{S}_z^2 + \Lambda_o(\hat{S}_+^4 + \hat{S}_-^4) \tag{1}$$

where $K_o, \Lambda_o > 0$.

(i) Find the energy levels of the problem when $\Lambda_o = 0$

(ii) Suppose now that $\Lambda_o > 0$, but $g_o = S^3\Lambda_o/K_o \ll 1$. Show that if S has an integer value, then pairs of levels which have projection $S_z = \pm M$, where M is an even number, are split by the term 4th-order in $\hat{S}_\pm$, with a splitting given to leading order in perturbation theory by

$$\Delta_M^S \;=\; A_M^S K_o \left(\frac{\Lambda_o}{K_o}\right)^{M/2} \quad ; \quad A_M^S \;=\; \frac{(-1)^{M/2\,-1}}{4^{M-2}[(M/2\;-1)!]^2}\,\frac{(S+M)!}{(S-M)!} \tag{2}$$

(iii) Now find an approximate expression for the ground-state splitting Δ_S^S when $S \gg 1$ and even; and also for the splitting Δ_M^S when $M \gg 1$ and $S - M \gg 1$, and both are even. Note that the latter expression does not tend to the former when $M \to S$.

(iv) The Hamiltonian in (i) can be diagonalised numerically very easily. Suppose do this for $S = 14$. Now, we wish to compare with both the perturbative result in (ii), for Δ_S^S, and with the large S result in (iii) for Δ_S^S. Plot all three results for $ln\Delta_S^S$ together as a function of (Λ_o/K_o).

QUESTION (2): SPIN INSTANTON

A spin tunneling in a biaxial potential with Hamiltonian $\mathcal{H} = -D\hat{S}_z^2 + E\hat{S}_x^2$, with $D, E > 0$. We will assume that $E \gg D$, so that, for large S, we are in the semiclassical regime where a WKB/instanton approximation is valid. Then, with this assumption, the semiclassical instanton paths for tunneling are approximately along the 2 lines, in the $\hat{y}\hat{z}$-plane, between the 'north' and 'south' poles of the Bloch sphere.

(i) Show that if we assume that $d\phi/d\tau \sim 0$ in the imaginary time equations of motion, then the instanton solutions are given, after eliminating ϕ, by

$$\sin\theta(\tau) \;=\; \frac{1}{\cosh(\omega_o\tau)} \quad ; \qquad \textit{where} \qquad \omega_o = 2S\sqrt{D(D+E)} \tag{3}$$

(ii) Then show that the semiclassical instanton action along these paths is given by the expression

$$\mathcal{S}_o^{(\eta)} \;=\; 2S\hbar \ln\left[\left(\frac{D+E}{E}\right)^{1/2} + \left(\frac{D}{E}\right)^{1/2}\right] \;+\; i\pi\eta S \tag{4}$$

where $\eta = \pm 1$ labels the paths.