Phys 501: HOMEWORK ASSIGNMENT No (1)

Friday January 14th 2011

DUE DATE: Friday Jan 28th 2011.

(Please note that assignments handed in late may not receive a full mark.)

QUESTION (1): CLASSICAL MECHANICS

(i) Consider a 2-dimensional classical harmonic oscillator, with Hamiltonian

$$\mathcal{H} \;=\; \frac{1}{2}\left[\left(\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}\right)+gx_1x_2+(m_1\omega_1x_1^2+m_2\omega_2x_2^2)\right] \tag{1}$$

Find the normal modes q_1, q_2 of this system, and the natural frequencies Ω_1, Ω_2.

(ii) The classical action $\mathcal{S}$ for a classical system can be written as $\mathcal{S} = \int(PdQ - \mathcal{H}dt)$, where P, Q are generalised momenta and coordinate variables, and $\mathcal{H}$ is the Hamiltonian. Derive the Hamilton-Jacobi equation for the action, in the form

$$\frac{\partial \mathcal{S}}{\partial t} + \mathcal{H}(\frac{\partial \mathcal{S}}{\partial Q}, Q; t) = 0 \tag{2}$$

and show that $\partial\mathcal{S}/\partial Q = P$. From this write down the Hamilton-Jacobi equation for a particle in a 2-d potential $V(\mathbf{r})$, and draw a picture which shows schematically the contours of constant action and the particle trajectories for the 2-dimensional harmonic oscillator, in terms of the original coordinates x_1, x_2, and their conjugate momenta.

QUESTION (2): DENSITY MATRICES

(i) Consider a 2-spin system of spin-1/2 particles, which has been prepared such that we have (a) a probability $1 - P = 1 - A^2$ that the system is in an incoherent mixture, with equal weighting, of states $|\psi_a\rangle = [|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle]/\sqrt{2}$ and $|\psi_b\rangle = [|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle]/\sqrt{2}$, and (b) a probability P the system is in pure state with wave-function $|\psi\rangle = [|\psi_a\rangle + |\psi_b\rangle]/\sqrt{2}$.

Find the total density matrix for this 2-spin system, quantizing along the $\hat{z}$-axis. Then calculate the expectation values of (a) the "up-up" operator $\hat{\sigma}_1^z\hat{\sigma}_2^z$, and (b) the "up-right" operator $\hat{\sigma}_1^z\hat{\sigma}_2^x$, for the pair of spins.

(ii) Now suppose we trace over spin 2, to get a reduced density matrix for spin 1. Find this reduced density matrix, and then calculate the expectation values of $\hat{\sigma}_1^z$, and $\hat{\sigma}_1^x$, for the first spin.

(iii) Comment on your results, focusing particularly on the way in which the degree to which the 2 spins are entangled appears in the resulting density matrices, and also on the way in which different measurements can reveal these properties.

QUESTION (3): PATH INTEGRALS

Suppose we have a simple 1-d harmonic oscillator driven by some arbitrary time-dependent force $F(t)$, with Lagrangian

$$\mathcal{L} \;=\; \frac{m}{2}(\dot{x}^2 - \omega_o^2x^2) - F(t)x \tag{3}$$

where x is the coordinate. We want to calculate transition amplitudes between the eigenstates of the SHO.

(i) Show that the "vacuum amplitude' $G_{oo}(t;F) = \langle 0|\hat{G}(t)|0\rangle$ from the ground state $|0\rangle$ of the free oscillator (with $F(t) = 0$), of energy $\hbar\omega_o$, at time 0, to the same state at time t, is given by

$$G_{oo}(t;F) \;=\; \exp\left[-\frac{1}{2m\hbar\omega_o}\int_0^t d\tau \int_0^{t'} d\tau' F(\tau)F(\tau')e^{i\omega_o(\tau-\tau')}\right] \tag{4}$$

(ii) Now see if you can find the amplitude $G_{10}(t)$ to go from $|0\rangle$ at time 0, to the lowest excited state $|1\rangle$, of energy $\frac{3}{2}\hbar\omega_o$, at time t.