PHYS 501

Solutions

HW#3

Q1) i)

$$|\psi\rangle = a|\uparrow\uparrow\rangle + be^{i\phi}|\downarrow\downarrow\rangle$$

$$\rho = |\psi\rangle\langle\psi| = \left(a|\uparrow\uparrow\rangle + be^{i\phi}|\downarrow\downarrow\rangle\right)\cdot\left(a^*\langle\uparrow\uparrow| + b^*e^{-i\phi}\langle\downarrow\downarrow|\right)$$

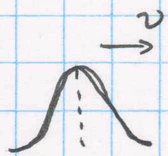
$$\Rightarrow \rho = |a|^2\,|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |b|^2\,|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + ab^*e^{-i\phi}\,|\uparrow\uparrow\rangle\langle\downarrow\downarrow| + a^*be^{i\phi}\,|\downarrow\downarrow\rangle\langle\uparrow\uparrow|$$

$$\Rightarrow \rho \doteq \begin{array}{c c} & \begin{array}{cccc} |\uparrow\uparrow\rangle & |\uparrow\downarrow\rangle & |\downarrow\uparrow\rangle & |\downarrow\downarrow\rangle \end{array} \\ \begin{array}{c} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{array} & \begin{pmatrix} |a|^2 & 0 & 0 & ab^*e^{-i\phi} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a^*be^{i\phi} & 0 & 0 & |b|^2 \end{pmatrix} \end{array}$$

$$Tr_2(\rho) = \sum_\beta \rho_{\alpha\beta\delta\beta}\,|\alpha\rangle\langle\delta| = |a|^2\,|\uparrow\rangle\langle\uparrow| + |b|^2\,|\downarrow\rangle\langle\downarrow|$$

$$\Rightarrow Tr_2(\rho) \doteq \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix}$$

ii) $$\hat{\mathcal{H}} = \frac{\hat{P}_0^2}{2m} + \sum_{j=1}^{N} V(x-x_j)\,\hat{\sigma}_j^x\,(1-\hat{\tau}_0^z)$$



$$G_{in}(x,t) = W(t)\, e^{\frac{-\pi(x-vt)^2}{W^2(t)}} e^{ik(x-vt)}$$

$$W(t=0) = W_0 \gg a_0$$

$$\iint V(x)\,dx\,dt = \pi/4$$

$$|\psi_0(x)\rangle = G_{in}(x,t) \times \left[a|\Uparrow\rangle + b e^{i\phi}|\Downarrow\rangle\right]$$

$$|\psi_{in}(x,t)\rangle = |\psi_0(x,t)\rangle \otimes |\uparrow\uparrow\uparrow\cdots\uparrow\rangle$$

$$e^{i\hat{\sigma}_x\beta} = \cos\beta\, \mathbb{1} + i\sin\beta\,\hat{\sigma}_x$$

$$|\psi_{out}(x,t)\rangle = e^{\frac{-i\hat{H}(t-t_0)}{\hbar}} |\psi_{in}(x,t)\rangle \qquad \text{assume } \hbar = 1$$

$$= e^{-i \hat{P}_0^2/2m\, \Delta t}\, e^{-i\sum_{j=1}^{N} V(x-x_j)\hat{\sigma}_j^x (1-\hat{\tau}_0^z)\Delta t} |\psi_{in}(x,t)\rangle$$

$$= e^{-i P_0^2/2m\,\Delta t}\, G_{in}(x,t)\left[a|\Uparrow\rangle \otimes |\uparrow\uparrow\uparrow\cdots\uparrow\rangle + b e^{i\phi}|\Downarrow\rangle \otimes e^{-2i\sum_{j=1}^{N}\int_{t_0}^{t} V(x-x_j)\hat{\sigma}_j^x\,dt} |\uparrow\uparrow\uparrow\cdots\uparrow\rangle\right]$$

$$e^{-2i\sum_{j=1}^{N}\int_{t_0}^{t} V(x-x_j)\hat{\sigma}_j^x dt} |\uparrow\uparrow\uparrow \cdots \uparrow\rangle$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\left(-2i\sum_{j=1}^{N}\int_{t_0}^{t} V(x-x_j)\,\hat{\sigma}_j^x\right)^m |\uparrow\uparrow\uparrow\cdots\uparrow\rangle$$

$$\left(\hat{\sigma}_j^x\right)^m = \begin{cases} \mathbb{1} & m \text{ even} \\ \hat{\sigma}_j^x & m \text{ odd} \end{cases}$$

$$\Rightarrow \quad = \cos\left[2\iint_{t_0}^{t} V(x')\,dx'\,dt'\right] |\uparrow\uparrow\cdots\uparrow\rangle$$

$$+ \sin\left[2\iint_{t_0}^{t} V(x')\,dx'\,dt'\right] |\downarrow\downarrow\downarrow\cdots\downarrow\rangle$$

Since $\iint V(x')\,dx'\,dt' = \pi/4 \Rightarrow e^{-2i\sum_{j=1}^{N}\int_{t_0}^{t} V(x-x_j)\hat{\sigma}_j^x dt} |\uparrow\uparrow\uparrow\cdots\uparrow\rangle$

$$= |\downarrow\downarrow\downarrow\downarrow\cdots\downarrow\rangle$$

$$\Rightarrow |\psi_{out}\rangle = G_{out}(x,t)\left[a\,|\Uparrow\rangle|\uparrow\uparrow\cdots\uparrow\rangle + b e^{i\phi}|\Downarrow\rangle|\downarrow\downarrow\cdots\downarrow\rangle\right]$$

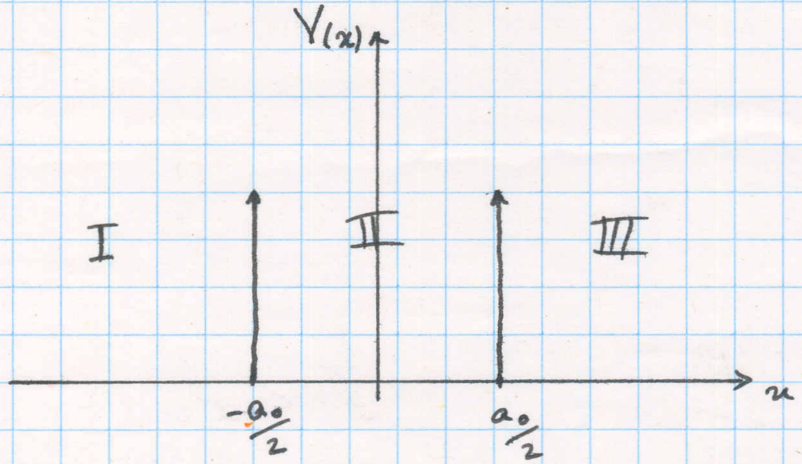
iii) It's a measurement of first kind since we have the original inital states of the system now entangled with the states of the apparatus $|\uparrow\uparrow\uparrow\cdots\uparrow\rangle$ and $|\downarrow\downarrow\downarrow\cdots\downarrow\rangle$ with the same coefficients a and $be^{i\varphi}$.

$$G_{out}(x,t) = G_{in}(x,t)$$

It means that the evolution of spatial part of the incoming wave ~~does~~ not been affected by the chain.

Q2)

$$V(x) = -V_0\left[\delta(x - a_0/2) + \delta(x + a_0/2)\right]$$



$$\left.\begin{array}{ll} \text{in Region } I: & \psi_I(x) = e^{ikx} + Re^{-ikx} \\ \text{" " } II: & \psi_{II}(x) = Ae^{ikx} + Be^{-ikx} \\ \text{" " } III: & \psi_{III}(x) = Te^{ikx} \end{array}\right\} \Leftarrow \psi(x) = \begin{cases} \psi_I(x) & x < -a_0/2 \\ \psi_{II}(x) & -a_0/2 < x < a_0/2 \\ \psi_{III}(x) & x > a_0/2 \end{cases}$$

Note that we chose above trial wavefunctions since $\hat{H} = P^2/2m$ in these regions.

Boundary conditions: $\left.\psi_I(x)\right|_{x = -a_0/2 - \varepsilon} = \left.\psi_{II}(x)\right|_{x = -a_0/2 + \varepsilon}$

$\varepsilon \to 0^+$

$$\psi_{II}(x)\Big|_{x=\frac{a_0}{2}-\varepsilon} = \psi_{III}(x)\Big|_{x=\frac{a_0}{2}+\varepsilon}$$

Since $V(x)$ consists of two delta functions, we have two discontinuity in first derivative of $\psi(x)$:

$$\int_{-\frac{a_0}{2}-\varepsilon}^{-\frac{a_0}{2}+\varepsilon} \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x) \right) dx$$

$$-\frac{\hbar^2}{2m}\left(\psi'_{II}(x)\Big|_{x=-\frac{a_0}{2}+\varepsilon} - \psi'_{I}(x)\Big|_{x=-\frac{a_0}{2}+\varepsilon} \right) = -\int_{-\frac{a_0}{2}-\varepsilon}^{-\frac{a_0}{2}+\varepsilon} V(x)\psi(x)\,dx = V_0\psi(x)\Big|_{x=-\frac{a_0}{2}}$$

$$\underset{\varepsilon \to 0^+}{\Longrightarrow} \quad \psi'_{II}(x)\Big|_{x=-\frac{a_0}{2}+\varepsilon} = \psi'_{I}(x)\Big|_{x=-\frac{a_0}{2}-\varepsilon} + \frac{2mV_0}{\hbar^2}\psi\left(x=-\frac{a_0}{2}\right)$$

and Similarly at $x=\frac{a_0}{2}$:

$$\psi'_{III}(x)\Big|_{x=\frac{a_0}{2}+\varepsilon} = \psi'_{II}(x)\Big|_{x=-\frac{a_0}{2}-\varepsilon} + \frac{2mV_0}{\hbar^2}\psi\left(x=\frac{a_0}{2}\right)$$

$$\varepsilon \to 0^+$$

So we 4 equations for 4 variables, R, T, A, B:

$$e^{\frac{-ika_0}{2}} + Re^{\frac{ika_0}{2}} - Ae^{\frac{-ika_0}{2}} - Be^{\frac{ika_0}{2}} = 0$$

$$ik\left[Ae^{\frac{-ika_0}{2}} - Be^{\frac{ika_0}{2}} - e^{\frac{-ika_0}{2}} + Re^{\frac{ika_0}{2}}\right] + \frac{2mV_0}{\hbar^2}\left(e^{\frac{-ika_0}{2}} + Re^{\frac{ika_0}{2}}\right) = 0$$

$$Ae^{\frac{ika_0}{2}} + Be^{\frac{-ika_0}{2}} - Te^{\frac{ika_0}{2}} = 0$$

$$ik\left[Te^{\frac{ika_0}{2}} - Ae^{\frac{ika_0}{2}} + Be^{\frac{-ika_0}{2}}\right] + \frac{2mV_0}{\hbar^2} Te^{\frac{ika_0}{2}} = 0$$

with some straightforward calculations we can solve the set of above equations to obtain R, A, B and T:

$$R = i\left(\frac{k_0}{k}\right) \cdot \frac{\cos ka_0 - \frac{k_0}{k}\sin ka_0}{\left(\frac{k_0}{k}\right)^2 e^{2ika_0} + \left(1 - \frac{ik_0}{k}\right)^2}$$

$$A = \frac{1 - i\left(\frac{k_0}{k}\right)}{\left(\frac{k_0}{k}\right)^2 e^{2ika_0} + \left(1 - \frac{ik_0}{k}\right)^2}$$

$$B = i\left(\frac{k_0}{k}\right) \cdot \frac{e^{ika_0}}{\left(\frac{k_0}{k}\right)^2 e^{2ika_0} + \left(1 - \frac{ik_0}{k}\right)^2}$$

$$T = \frac{1}{\left(\frac{k_0}{k}\right)^2 e^{2ika_0} + \left(1 - \frac{ik_0}{k}\right)^2}$$

where $k_0 \equiv \dfrac{mV_0}{\hbar^2}$.

As it's obvious in the Fig.1, $|R|^2 + |T|^2 = 1$.

Q3) i)

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} - \frac{e\vec{A}}{c}\right)^2 + V(r)$$

$$V(r) = \begin{cases} V_0 & r<R \\ 0 & r>R \end{cases}$$

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r} = \frac{1}{2}\frac{\Phi}{\pi a_0^2}\Theta(a_0-r)\underbrace{\hat{z}\times\vec{r}}_{r\hat{\varphi}} \qquad r<a_0$$

$$\Rightarrow \vec{A} = \frac{\Phi r}{2\pi a_0^2}\Theta(a_0-r)\hat{\varphi} \qquad r<a_0 \;, \quad \vec{A} = \frac{\Phi}{2\pi r}\hat{\varphi} \quad r>a_0$$

then for $r<a_0$ we get -

$$\mathcal{H} = -\frac{\hbar^2}{2m}\left(\vec{\nabla} - \frac{ie\vec{A}}{\hbar c}\right)^2 + V_0$$

$$\mathcal{H} = -\frac{\hbar^2}{2m}\left(\vec{\nabla} - \frac{ie\Phi r}{(\hbar c)2\pi a_0^2}\hat{\varphi}\right)^2 + V_0$$

$$= -\frac{\hbar^2}{2m}\left(\frac{1}{r}\partial_r(r\partial_r) + \left(\frac{1}{r}\frac{\partial}{\partial\varphi} - \frac{i\Phi r}{\Phi_0 a_0^2}\right)^2\right) + V_0$$

$$\left(\frac{1}{r}\frac{\partial}{\partial\varphi}-i\frac{\Phi}{\Phi_0}\frac{r}{a_0^2}\right)^2=\frac{1}{r^2}\left(\frac{\partial}{\partial\varphi}-i\alpha\left(\frac{r}{a_0}\right)^2\right)^2 \qquad r<a_0$$

$$=\frac{1}{r^2}\left(\partial_\varphi^2-2i\alpha\left(\frac{r}{a_0}\right)^2\frac{\partial}{\partial\varphi}-\alpha^2\left(\frac{r}{a_0}\right)^4\right)$$

for $r>a_0$

$$\left(\frac{1}{r}\frac{\partial}{\partial\varphi}-i\frac{\Phi}{\Phi_0}\cdot\frac{1}{r}\right)^2=\frac{1}{r^2}\left(\frac{\partial}{\partial\varphi}-i\alpha\right)^2$$

Therefore we have:

$$\mathcal{H}=\begin{cases}-\frac{\hbar^2}{2m}\left[r^{-1}\partial_r(r\partial_r)+r^{-2}\partial_\varphi^2-\frac{2i\alpha}{a_0^2}\frac{\partial}{\partial\varphi}-\frac{\alpha^2}{a_0^4}r^2\right]+V(r) & r<a_0\\[2em] -\frac{\hbar^2}{2m}\left[r^{-1}\partial_r(r\partial_r)+r^{-2}(\partial_\varphi-i\alpha)^2\right]+V(r) & r\geqslant a_0\end{cases}$$

ii) Assuming $a_0\longrightarrow 0$ we have

$$\mathcal{H}=-\frac{\hbar^2}{2m}\left[r^{-1}\partial_r(r\partial_r)+r^{-2}(\partial_\varphi-i\alpha)^2\right]+V(r) \qquad \text{where} \quad V(r)=\begin{cases}0 & r>R\\ V_0 & r<R\end{cases}$$

$$\mathcal{H}\,\Psi(\vec{r})=E\,\Psi(\vec{r})$$

$$\psi(\vec{r}) = R(r)\, P(\varphi)$$

$$\Rightarrow \; r^{-1} P(\varphi)\, d_r(r\, d_r R(r)) + r^{-2} R(r) (\partial_\varphi - i\alpha)^2 P(\varphi)$$

$$+ \underbrace{\frac{2m}{\hbar^2}(E - V(r))}_{k^2 = \begin{cases} k_I^2 & \text{for } r<R \\ k_{II}^2 & \text{for } r>R \end{cases}} R(r)\, P(\varphi) = 0$$

$$\Rightarrow \; \frac{r}{R(r)} \frac{d}{dr}\left(r \frac{dR(r)}{dr}\right) + \frac{1}{P(\varphi)} (\partial_\varphi - i\alpha)^2 P(\varphi) + (kr)^2 = 0$$

$$\frac{1}{P(\varphi)} (\partial_\varphi - i\alpha)^2 P(\varphi) = -\ell^2$$

$$\Rightarrow \; P(\varphi) = e^{i(\pm\ell + \alpha)\varphi}$$

Since the wavefunction must be well-defined $\psi(r, \varphi + 2\pi) = \psi(r, \varphi)$ $\ell + \alpha$ must be an integer.

$$\ell + \alpha = m \qquad m \in \mathbb{Z} \;\Rightarrow\; \ell = \underbrace{\pm\alpha, \pm 1 \pm \alpha, \pm 2 \pm \alpha, \pm 3 \pm \alpha, \ldots}_{\equiv L}$$

$$\frac{r}{R(r)} \frac{d}{dr}\left(r \frac{dR(r)}{dr}\right) - \ell^2 + (kr)^2 = 0$$

$$\Rightarrow \; r^2 \frac{d^2 R(r)}{dr^2} + r \frac{dR(r)}{dr} + \left[(kr)^2 - \ell^2\right] R(r) = 0$$

$$R(\bar{r}) \Rightarrow R(kr) \;\Rightarrow\; \bar{r}^2 R''(\bar{r}) + \bar{r} R'(\bar{r}) + \left[\bar{r}^2 - \ell^2\right] R(\bar{r}) = 0$$

$\bar{r} = kr$

Bessel's differential Equation.

$E > V_0$ case:

$$k^2 = \begin{cases} k^2 > 0 & r < R \\ k_0^2 = \frac{2mE}{\hbar^2} & r > R \end{cases} \quad \text{where} \quad k^2 = \frac{2m}{\hbar^2}(E - V_0)$$

in 2d:

$$\psi(r,\varphi) \underset{kr \gg 1}{\approx} \underbrace{e^{i k \cdot r}}_{\text{initial wave}} + \underbrace{\frac{f_k(\theta)}{\sqrt{r}} e^{i(kr + \frac{\pi}{4})}}_{\text{scattered part}}$$

$$\Rightarrow \psi(r,\varphi) \underset{kr \gg 1}{\longrightarrow} \left(\frac{2}{\pi k r}\right)^{1/2} \sum_{l} A_l \, i^l e^{il\phi} \cos\left(kr - \frac{\pi}{2}\left(l + \frac{1}{2}\right) + \delta_l\right)$$

where $A_l = e^{i\delta_l}$

$$\Rightarrow \psi(r,\varphi) \longrightarrow \left(\frac{2}{\pi k r}\right)^{1/2} \sum_{l} e^{i\delta_l} i^l e^{il\varphi} \left[\cos\left(kr - \frac{\pi l}{2} - \frac{\pi}{4}\right)\cos\delta_l - \sin\left(kr - \frac{\pi l}{2} - \frac{\pi}{4}\right)\sin\delta_l\right]$$

from the asymptotic form of Bessel functions we get:

$$J_l(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{l\pi}{2} - \frac{\pi}{4}\right) , \quad Y_l(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{l\pi}{2} - \frac{\pi}{4}\right)$$

Since Bessel functions $J_l(x)$, $Y_l(x)$ form a complete set of functions we can write $\psi_{outside}(r,\varphi)$ in the following form:

$$\underset{r > R}{\psi(r,\varphi)} = \sum_{l \in \mathbb{Z}} e^{i\delta_l} i^l e^{il\varphi} \left[\cos\delta_l J_l(k_0 r) - \sin\delta_l Y_l(k_0 r)\right]$$

for $r<R$ we have

$$\Psi(r,\varphi) = \sum_{l\in\mathbb{Z}} a_l e^{il\varphi} J_{|l+\alpha|}(kr)$$

above form is obtained by recalling that $\Psi(r,\varphi)$ must be finite for $r \longrightarrow 0$.

Therefore by considering the continuity of $\Psi(r,\varphi)$ at $r=R$ and also the continuity of it's derivative, we can obtain all $\delta_m(k,k_0)$ $m\in\mathbb{Z}$.

$$\left.\frac{1}{R(kr)}\frac{dR(kr)}{dr}\right|_{r=R^+} = \left.\frac{1}{R(kr)}\frac{dR(kr)}{dr}\right|_{r=R^-}$$

after some straightforward algebra we get:

$\Longrightarrow$

$$\delta_m = \tan^{-1}\left[\frac{J'_{|m+\alpha|}(kR)\,Y_m(k_0R) - \frac{k_0}{k} J_{|m+\alpha|}(kR)\,Y'_m(k_0R)}{J'_{|m+\alpha|}(kR)\,J_m(k_0R) - \frac{k_0}{k} J_{|m+\alpha|}(kR)\,J'_m(k_0R)}\right]$$

for $0<E<V_0$ case one should use modified Bessel functions of first kind i.e, $I_{|m+\alpha|}(kR)$ instead of $J_{|m+\alpha|}(kR)$ and similarly we can obtain δ_m in this case.

iii)

$$G(r_2\theta_2;r_1\theta_1;t) = \int \sum_{l\in\mathbb{Z}} e^{-iE_k t}\psi_{lk}(r_2,\theta_2)\psi^*_{lk}(r_1,\theta_1)\, d^2k$$

$$\psi_{lk}(r,\theta) = \begin{cases} e^{i\delta_l} i^l e^{il\theta}\left[\cos\delta_l J_l(kr) - \sin\delta_l Y_l(kr)\right] & r>R \\ a_l e^{il\theta} J_{|l+\alpha|}(k'r), \text{ where } k'^2 = \frac{2m}{\hbar^2}(E-V_0) & r>R \end{cases}$$

for $V_0 \longrightarrow \infty$, $R \longrightarrow 0$ $\Rightarrow$ $\psi_{lk}(r,\theta) = 0$ $\quad r\to 0$

$$\Rightarrow \sin\delta_l(k,k') = 0 \Rightarrow \delta_l = n\pi$$

$$\Rightarrow G(r_2\theta_2;r_1\theta_1;t) = \sum_{l=-\infty}^{\infty} \int J^*_l(kr_1) J_l(kr_2) e^{-iE_k t} d^2k\, e^{il(\theta_2-\theta_1)}$$

The end of HW#3.