P501: SOLUTION to ASSIGNMENT 1

1. **(i)** In the σ_z basis, with probability 1/2 we have $[1, i]^T$, which in density matrix form is

$$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}; \tag{1}$$

With probability 1/2 we also have

$$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2}$$

Therefore, the total density matrix is

$$\rho = \frac{1}{2} \times \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} + \frac{1}{2} \times \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{i}{4} \\ \frac{i}{4} & \frac{1}{2} \end{bmatrix}. \tag{3}$$

Similarly in the σ_y basis,

$$\rho = \frac{1}{2} \times \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \times \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}. \tag{4}$$

Notice that in any basis, the density matrix of the completely incoherent state, with half spin-up and half spin-down, is the same.

(ii): The expectation values $\langle A \rangle = Tr\,(A\rho)$, are $\langle \sigma_x \rangle = 0$; $\langle \sigma_y \rangle = 0.5$, and $\langle \sigma_z \rangle = 0$ respectively.

2. **(i)** To avoid large matrices, we can work with the bra-ket notation. The total density matrix is then

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| = \frac{1}{3}\Big(&|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + e^{-i\chi}|\uparrow\uparrow\rangle\langle\downarrow\uparrow| + e^{-i\phi}|\uparrow\uparrow\rangle\langle\downarrow\downarrow| + \\ &e^{i\chi}|\downarrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + e^{i(\chi-\phi)}|\downarrow\uparrow\rangle\langle\downarrow\downarrow| + \\ &e^{i\phi}|\downarrow\downarrow\rangle\langle\uparrow\uparrow| + e^{i(\phi-\chi)}|\downarrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\Big).\end{aligned} \tag{5}$$

(ii) Then the reduced density matrix for the first spin is

$$\rho_I = Tr_{II}\,(\rho) = \frac{1}{3}\begin{bmatrix} 1 & e^{-i\chi} \\ e^{i\chi} & 2 \end{bmatrix}. \tag{6}$$

Here $Tr_{II}\,(\cdot)$ means a partial trace on the second system.

(iii) The expectation values are $\langle \sigma_1^x \rangle = \frac{2}{3}\cos\chi$ and $\langle \sigma_1^z \rangle = -\frac{1}{3}$ respectively.