Phys 501: HOMEWORK ASSIGNMENT No (3)

Wednesday March 28th 2012

DUE DATE: Wednesday April 4th 2012.

NB: Please note that assignments handed in late may not receive a full mark.

QUESTION (1): ADIABATIC PROCESSES

(i) The time-dependent Schrodinger equation $\mathcal{H}(t)\Psi(t) = i\hbar\partial_t\Psi(t)$. In the limit where $\mathcal{H}(t)$ changes slowly as a function of some parameter $\mathbf{g}(t)$, we write $\mathcal{H}(t)\psi_n(t) = E_n(t)\psi_n(t)$, where now t is simply a fixed parameter; and we write $\Psi(t)$ in the form

$$\Psi(t) \;=\; \sum_n c_n(t)\psi_n(t)e^{\frac{-i}{\hbar}\int^t dt' E_n(t')} \tag{1}$$

(i) First show that

$$\langle\psi_n|\partial_t|\psi_m\rangle \;=\; -\frac{\langle\psi_n|\partial_t\mathcal{H}|\psi_m\rangle}{(E_n(t)-E_m(t))} \tag{2}$$

when $m \neq n$. Then, consider the case where $m = n$. We imagine taking the Hamiltonian slowly around a circuit, by varying $\mathbf{g}(t)$ over a long time period so as to bring it back to its original value. If the Berry phase is defined as $\phi_B^n(\mathcal{C}) = i\oint_{\mathcal{C}} d\mathbf{g}\cdot\langle n(g)|\nabla_g n(g)\rangle$, where the line integral is taken around the circuit $\mathcal{C}$, then show that

$$\begin{aligned}\phi_B^n(\mathcal{C}) \;&=\; -Im\oint_{\mathcal{C}} d\mathbf{S}\cdot\nabla_g\times\langle n|\nabla_g n\rangle \\ &=\; -Im\oint_{\mathcal{C}} d\mathbf{S}\cdot\sum_{m\neq n}\langle\nabla_g n|m\rangle\times\langle m|\nabla_g n\rangle\end{aligned} \tag{3}$$

where the integration is over the surface in parameter space (ie., $\mathbf{g}$-space) enclosed by the closed curve $\mathcal{C}$.

(ii) Consider 2 potential wells (shaped however you want - eg., harmonic oscillators, or attractive Coulomb potentials), with centres situated at positions $x = \pm R_o$. Show how you think the energy levels will vary as a function of R_o in a graph.

QUESTION (2): SPIN DYNAMICS

We first consider a spin-1/2 system with the simple Hamiltonian $\mathcal{H} = \gamma\mathbf{B}(t)\cdot\hat{\boldsymbol{\sigma}}$, where $\hat{\boldsymbol{\sigma}}$ is the Pauli vector, γ is a constant, and $\mathbf{B}(t)$ is a magnetic field with some arbitrary time-dependence.

(i) Consider the above Hamiltonian where $\mathbf{B}(t)$ is the time-independent constant $\mathbf{B} = \hat{x}\Delta + \hat{z}\epsilon$. Find the amplitude $G_{\uparrow\uparrow}(\tau)$ for the spin to start off at time $t = 0$ in state $|\uparrow\rangle$, and end up at time $t = \tau$ in the same state $|\uparrow\rangle$.

(ii) Again assuming that $\mathbf{B} = \hat{x}\Delta + \hat{z}\epsilon$, find $G_{\uparrow\uparrow}(\tau)$ as a series expansion in $\Delta\tau$, by expanding the expression $G_{\uparrow\uparrow}(\tau) = \langle\uparrow|exp[-i\mathcal{H}\tau/\hbar]|\uparrow\rangle$. Then sum this series in the particular case where $\epsilon = 0$.

(iii) Now let us consider a spin $S \gg 1$, and suppose it has the spin Hamiltonian

$$\mathcal{H} \;=\; \hbar\left(\frac{k_2^z}{S}\hat{S}_z^2 + \frac{1}{S^3}[k_4^z\hat{S}_z^4 + \frac{1}{2}k_4^{\perp}(\hat{S}_+^4 + \hat{S}_-^4)] + \frac{k_6}{2S^5}\hat{S}_z^2(\hat{S}_+^4 + \hat{S}_-^4)]\right) \tag{4}$$

Are there any restrictions on the value of S implied in this Hamiltonian?

Write down the Lagrangian for this system, in terms of the classical spin vector $\mathbf{S} = S\mathbf{n}$, where $\mathbf{n}$ is a unit vector on the Bloch sphere. Then show a topographical map on the Bloch sphere of the spin potential given in this equation, assuming that all of the constants coefficients in the Hamiltonian are positive except for k_2^z.

Finally, find the equation of motion for the spin components of the classical vector $\mathbf{S}$, for this potential.