Phys 501: HOMEWORK ASSIGNMENT No (2)

Friday January 22nd 2010

DUE DATE: Wednesday Feb 3rd 2010.
Assignments handed in late may not receive a full mark.

QUESTION (1): PROBLEMS WITH PROPAGATORS

(i) The propagator for a particle in quantum mechanics is the unitary operator producing time evolution of the wave-function, ie.,

$$\hat{G}(t_2 - t_1) = e^{-\frac{i}{\hbar}\hat{H}(t_2 - t_1)} \tag{1}$$

Show that this operator also satisfies the defining equation for a Green function, viz., that

$$(\hat{H} - i\hbar\partial_t)\hat{G}(t - t') \;=\; -i\hbar\hat{\mathbf{1}}\delta(t - t') \tag{2}$$

where $\hat{\mathbf{1}}$ is the unit operator.

(ii) Consider a free particle of mass m, moving in one dimension. Assuming it moves between spacetime points x_1, t_1 and x_2, t_2, find the classical action $\mathcal{S}_c$ for the classical path between these points. The quantum propagator for the same boundary conditions is $G(x_2 - x_1, t_2 - t_1) = A_o e^{i\mathcal{S}_c/\hbar}$, where we need to find A_o. Noting that G is diagonal in the momentum basis for this free particle, evaluate it and thereby find the prefactor A_o.

(iii) Now suppose the particle is moving inside a box, ie., it is confined to the region $-L < x < L$ by an infinite potential for $|x| > L$. Find $G(x_2 - x_1, t_2 - t_1)$, as a series expansion over the eigenfunctions of the system.

(iv) Now consider the problem where the particle moves in a linear potential $V(x) = -\alpha x$. What is the classical equation of motion for this system? Find the quantum propagator $G(x_2 - x_1, t_2 - t_1)$ for this system, by first finding the classical action as in (ii) above, and then show that the fluctuation prefactor A_o is unchanged from that for the free particle.

QUESTION (2): AVERAGING OVER NOISE

(i) Now consider the problem of a free particle in a 'noise field' $F(t)$, for which the Lagrangian is

$$L \;=\; \frac{m}{2}\dot{x}^2 - F(t)x$$

Suppose that at time t_1 the system is at x_1, and at time t_2 it is at x_2. Show that the propagator between these 2 points is $G(x_2, x_1; t_2, t_1|F) = A^o_{2,1} e^{iS_{21}/\hbar}$, in which A^o_{21} is the prefactor for a free particle, and

$$\begin{aligned} S_{21} \;=\; &= x_2 \int_{t_1}^{t_2} d\tau F(\tau) - \frac{1}{2m}\int_{t_1}^{t_2} d\tau \left[\int_{t_1}^{\tau} d\tau' F(\tau')\right]^2 \\ &+ \frac{m}{2}\frac{1}{(t_2 - t_1)}\left[(x_2 - x_1) - \frac{1}{m}\int_{t_1}^{t_2} d\tau \int_{t_1}^{\tau} F(\tau')d\tau'\right]^2 \end{aligned} \tag{3}$$

in which $x_1 = x(t_1)$, and $x_2 = x(t_2)$ (the latter produced by simple integration of the equations of motion).

(ii) Now suppose that the noise function $F(t)$ is white noise, with a correlator $\langle F(t)F(t')\rangle = \alpha\delta(t - t')$, where α is a constant. Find the noise-averaged propagator $\langle G(x_2, x_1; t_2, t_1|F)\rangle_F$ for a free particle exposed to this white noise.