Q1: (i) $$\hat{G}(t_2-t_1) = \begin{cases} e^{\frac{-i}{\hbar}\hat{H}(t_2-t_1)} & t_2 \geqslant t_1 \\ 0 & t_2 < t_1 \end{cases}$$

Above propagator is for a time-independent Hamiltonian.

for $t \neq t' \Rightarrow \begin{cases} t>t' & i\hbar \frac{\partial}{\partial t} G(t-t') = i\hbar \times \frac{-i}{\hbar} \hat{H} e^{\frac{-i}{\hbar}\hat{H}(t-t')} = \hat{H} G(t-t') \\ t<t' & i\hbar \frac{\partial}{\partial t} G(t-t') = 0 = \hat{H} 0 = \hat{H} G(t-t') \end{cases}$

$\Rightarrow$ for $t \neq t'$ this operator satisfies the defining equation given in the question.

for $t-t' \to 0$ $\quad \hat{G}(t-t') = e^{\frac{-i}{\hbar}\hat{H}(t-t')} \theta(t-t')$ where $\theta(x) = \begin{cases} 1 & x \geqslant 0 \\ 0 & x<0 \end{cases}$

$$i\hbar \frac{\partial}{\partial t} \hat{G}(t-t') = \hat{H} e^{\frac{-i}{\hbar}\hat{H}(t-t')} + e^{\frac{-i}{\hbar}\hat{H}(t-t')} \cdot i\hbar \frac{\partial}{\partial t} \theta(t-t')$$

$$\frac{\partial}{\partial t}\theta(t-t') = \delta(t-t') \Rightarrow i\hbar \partial_t \hat{G}(t-t') = \hat{H} G(t-t') + i\hbar \underbrace{e^{\frac{-i}{\hbar}\hat{H}(t-t)}}_{e^0 = \mathbb{1}} \delta(t-t')$$

$$\Rightarrow i\hbar \partial_t \hat{G}(t-t') = \hat{H} G(t-t') + i\hbar \mathbb{1} \delta(t-t')$$

$$\Rightarrow (\hat{H} - i\hbar \partial_t) \hat{G}(t-t') = -i\hbar \mathbb{1} \delta(t-t')$$

(ii) $$\mathcal{L} = \frac{1}{2} m \dot{x}^2$$

$$S_{cl} = \int_{t_1}^{t_2} \mathcal{L}(t)\,dt = \frac{m}{2}\int_{t_1}^{t_2} \dot{x}^2\,dt$$

$$\dot{x}_{cl} = \dot{x}(t=t_1) = v_0 = \frac{x_2 - x_1}{t_2 - t_1}$$

$$\Rightarrow S_{cl} = \frac{m}{2} v_0^2 (t_2 - t_1) = \frac{m(x_2-x_1)^2}{2(t_2-t_1)}$$

$$G(x_2 - x_1; t_2 - t_1) = \langle x_2 | \hat{G}(t_2 - t_1) | x_1 \rangle$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle x_2 | p \rangle \langle p | \hat{G}(t_2-t_1) | p' \rangle \langle p' | x_1 \rangle \, dp\, dp'$$

$\hat{G}(t_2 - t_1)$ is diagonal in the momentum basis

$$= \langle p | \hat{G}(t_2 - t_1) | p' \rangle = \delta(p - p')\, e^{-\frac{i}{\hbar}\frac{p^2(t_2-t_1)}{2m}}$$

$$\Rightarrow G(x_2 - x_1, t_2 - t_1) = \frac{1}{(2\pi\hbar)} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}p(x_2-x_1) - \frac{i}{\hbar}\frac{p^2(t_2-t_1)}{2m}}\, dp$$

$$\int_{-\infty}^{\infty} e^{-Au^2+Bu}\,du = \sqrt{\frac{\pi}{A}}\, e^{B^2/4A} \qquad A>0$$

$$\left.\begin{aligned} A &= \frac{i(t_2-t_1)}{2m\hbar} \\ B &= \frac{i}{\hbar}(x_2-x_1) \end{aligned}\right\} \Rightarrow G(x_2-x_1, t_2-t_1) = \frac{1}{2\pi\hbar}\cdot\left(\frac{\pi}{\frac{i(t_2-t_1)}{2m\hbar}}\right)^{1/2} e^{\frac{im(x_2-x_1)^2}{2(t_2-t_1)}}$$

$$\Rightarrow G(x_2-x_1, t_2-t_1) = \underbrace{\left[\frac{m}{2\pi i(t_2-t_1)}\right]^{1/2}}_{A_0} e^{\frac{im(x_2-x_1)^2}{2(t_2-t_1)}}$$

$$\Rightarrow \quad G(x_2-x_1, t_2-t_1) = A_0\, e^{iS_{cl}/\hbar}$$

(iii) $$G(x_2-x_1, t_2-t_1) = \langle x_2|\hat{G}(t_2-t_1)|x_1\rangle$$

$$= \sum_n \langle x_2|n\rangle\, e^{-\frac{i}{\hbar}E_n(t_2-t_1)} \langle n|x_1\rangle$$

$$\langle x|n\rangle = \phi_n(x) = \frac{1}{\sqrt{L}}\begin{cases} \sin\left(\frac{n\pi x}{2L}\right) & n \text{ even} \in \mathbb{N} \\ \cos\left(\frac{n\pi x}{2L}\right) & n \text{ odd} \in \mathbb{N} \end{cases}$$

$$G(x_2-x_1, t_2-t_1) = \frac{1}{L}\sum_{l=1}^{\infty} \sin\left(\frac{l\pi x_2}{L}\right)\sin\left(\frac{l\pi x_1}{L}\right) e^{\frac{-il^2\pi^2\hbar}{L^2}(t_2-t_1)}$$

$$+ \frac{1}{L}\sum_{l=1}^{\infty} \cos\left((l-\tfrac{1}{2})\frac{\pi x_2}{L}\right)\cos\left((l-\tfrac{1}{2})\frac{\pi x_1}{L}\right) e^{\frac{-i(l-\frac{1}{2})^2\pi^2\hbar(t_2-t_1)}{L^2}}$$

(iv) $L = \frac{1}{2}m\dot{x}^2 + \alpha x$

$$\Rightarrow S_{cl} = \int_{x_1,t_1}^{x_2,t_2} L\,dt = \frac{1}{2}m\int \dot{x}^2(t)\,dt + \alpha\int x(t)\,dt$$

$$\dot{x}^2(t) = \frac{d}{dt}\left(x(t)\dot{x}(t)\right) - x(t)\ddot{x}(t)$$

$$m\ddot{x}(t) = \alpha \Rightarrow \frac{1}{2}m\dot{x}^2(t) = \frac{1}{2}m\frac{d}{dt}\left(x(t)\dot{x}(t)\right) - \frac{\alpha x(t)}{2}$$

$$\Rightarrow S_{cl} = \frac{1}{2}m\int_{t_1}^{t_2}\frac{d}{dt}\left(x(t)\dot{x}(t)\right) + \frac{\alpha}{2}\int_{t_1}^{t_2} x(t)\,dt$$

$$x(t) = \frac{\alpha}{2m}(t-t_1)^2 + v_0(t-t_1) + x_1$$

$$\Longrightarrow S_{cl} = \frac{1}{2} m \left[x_2 v_2 - x_1 v_1 \right] + \frac{\alpha}{2} \int_{t_1}^{t_2} \left[\frac{\alpha}{2m}(t-t_1)^2 + v_0 (t-t_1) + x_1 \right] dt$$

$$\Longrightarrow S_{cl} = \frac{m}{2}(x_2 v_2 - x_1 v_1) + \frac{\alpha^2}{12m}(t_2-t_1)^3 + \frac{\alpha v_0}{4}(t_2-t_1)^2 + \frac{\alpha x_1 (t_2-t_1)}{2}$$

$$\dot{x}(t) = \frac{\alpha}{m}(t-t_1) + v_0$$

$$\begin{cases} v_1 = v_0 \\ v_2 = \dfrac{\alpha(t_2-t_1)}{m} + v_0 \end{cases} \qquad v_2^2 - v_1^2 = \frac{2\alpha}{m}(x_2-x_1)$$

$$\Longrightarrow \frac{\alpha^2 (t_2-t_1)^2}{m^2} + \frac{2\alpha v_0 (t_2-t_1)}{m} = \frac{2\alpha(x_2-x_1)}{m}$$

$$v_0 = \frac{(x_2-x_1)}{(t_2-t_1)} - \frac{\alpha(t_2-t_1)}{2m}$$

$$\Longrightarrow \begin{cases} v_1 = \dfrac{x_2-x_1}{t_2-t_1} - \dfrac{\alpha(t_2-t_1)}{2m} \\ v_2 = \dfrac{x_2-x_1}{t_2-t_1} + \dfrac{\alpha(t_2-t_1)}{2m} \end{cases} \Longrightarrow$$

by plugging in these two eqs into S_{cl} equation, we obtain S_{cl} as a function of only x_2, t_2, x_1, t_1.

$$x(t) = x_{cl}(t) + q(t)$$

$$\Rightarrow S = S_{cl} + \delta S = S_{cl} + \int_{t_1,\, q(t_1)=0}^{t_2,\, q(t_2)=0} \left(m \dot{x} \dot{q} + \tfrac{1}{2} m \dot{q}^2 + \alpha q \right) dt$$

$$\dot{x} \dot{q} = \frac{d}{dt}(q \dot{x}) - q \ddot{x}$$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \Big(\underbrace{\cancel{m \frac{d}{dt}(q\dot{x})}}_{0} - \cancel{\underbrace{m\ddot{x}}_{\alpha} q + \alpha q} + \tfrac{1}{2} m \dot{q}^2 \Big) dt$$

Since $q(t_1) = q(t_2) = 0$

$$\Rightarrow \delta S = \int_{t_1,\, q(t_1)=0}^{t_2,\, q(t_2)=0} \tfrac{1}{2} m \dot{q}^2 \, dt \longrightarrow \text{free particle action}$$

$$\Rightarrow G(x_2, t_2; x_1, t_1) = A_{\text{free-particle}} \, e^{iS_{cl}/\hbar}$$

$$A_{\text{free-particle}} = \left[\frac{m}{2\pi i (t_2 - t_1) \hbar} \right]^{1/2}$$

Q2)

$$L = \frac{m}{2}\dot{x}^2 - xF(t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} + F(t) = 0$$

$$S_{cl} = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \left(\frac{m}{2}\dot{x}^2 - xF(t)\right)dt$$

$$-xF(t) = mx\ddot{x} = m\frac{d}{dt}(x\dot{x}) - m\dot{x}^2$$

$$\Rightarrow S_{cl} = \int_{t_1}^{t_2}\left(-\frac{m}{2}\dot{x}^2\right)dt + m\left(x_2\dot{x}_2 - x_1\dot{x}_1\right)$$

$$\dot{x}(t) = -\frac{1}{m}\int_{t_1}^{t} F(\tau)\,d\tau + \dot{x}_1$$

$$\dot{x}_2 = -\frac{1}{m}\int_{t_1}^{t_2} F(\tau)\,d\tau + \dot{x}_1$$

$$\dot{x}_2 x_2 - \dot{x}_1 x_1 = -\frac{x_2}{m}\int_{t_1}^{t_2} F(\tau)\,d\tau + \dot{x}_1(x_2 - x_1)$$

$$\Rightarrow m(\dot{x}_2 x_2 - \dot{x}_1 x_1) = -x_2\int_{t_1}^{t_2} F(\tau)\,d\tau + m\dot{x}_1(x_2 - x_1)$$

$$-\frac{m\dot{x}^2}{2} = -\frac{1}{2m}\left[\int_{t_1}^{t} F(\tau')d\tau'\right]^2 - \frac{m\dot{x}_1^2}{2} + \dot{x}_1\int_{t_1}^{t} F(\tau')d\tau'$$

$$\dot{x}_1 = \frac{x_2-x_1}{t_2-t_1} + \frac{1}{m(t_2-t_1)}\int_{t_1}^{t_2} d\tau \int_{t_1}^{\tau} d\tau' F(\tau')$$

$$-\frac{m}{2}\int_{t_1}^{t_2} d\tau\, \dot{x}^2 = -\frac{1}{2m}\int_{t_1}^{t_2} d\tau \left[\int_{t_1}^{\tau} F(\tau')d\tau'\right]^2 + \dot{x}_1\int_{t_1}^{t_2} d\tau \int_{t_1}^{\tau} F(\tau')d\tau'$$

$$-\frac{m}{2}\dot{x}_1^2(t_2-t_1)$$

$$\int_{t_1}^{t_2} d\tau \int_{t_1}^{\tau} F(\tau')d\tau' = \left(\dot{x}_1 - \frac{x_2-x_1}{t_2-t_1}\right) m(t_2-t_1)$$

$$\Rightarrow -\frac{m}{2}\int_{t_1}^{t_2} d\tau\, \dot{x}^2 = -\frac{1}{2m}\int_{t_1}^{t_2} d\tau \left[\int_{t_1}^{\tau} F(\tau')d\tau'\right]^2 + \frac{m}{2}\dot{x}_1^2(t_2-t_1) - m\dot{x}_1(x_2-x_1)$$

by plugging in $\dot{x}_1$ and adding up all the terms we get:

$$\Rightarrow S_{cl} = -x_2\int_{t_1}^{t_2} F(\tau)d\tau - \frac{1}{2m}\int_{t_1}^{t_2} d\tau \left[\int_{t_1}^{\tau} F(\tau')d\tau'\right]^2 + \frac{m}{2(t_2-t_1)}\left[(x_2-x_1) + \frac{1}{m}\int_{t_1}^{t_2} d\tau \int_{t_1}^{\tau} F(\tau')d\tau'\right]^2$$

(ii) $$\chi(t-t') = \langle F(t)F(t')\rangle = \alpha\,\delta(t-t')$$

$$\Rightarrow \quad P_G(F) = \frac{1}{\sqrt{\alpha}}\, e^{-\frac{1}{2\alpha}\int dt\, F^2(t)}$$

$$\Rightarrow \quad \langle G(x_2,x_1;t_2,t_1|F)\rangle_F = \frac{\int DF\, G\, e^{-\frac{1}{2\alpha}\int F^2(t)\,dt}}{\underbrace{\int DF\, e^{-\frac{1}{2\alpha}\int F^2(t)\,dt}}_{Z_0}}$$

$$G = \int Dx\, e^{\frac{i}{\hbar}\int\left(\frac{m\dot{x}^2}{2} - F(t)x\right)dt}$$

$$\Rightarrow \quad \langle G\rangle_F = \frac{1}{Z_0}\iint Dx\, DF\, e^{-\frac{1}{2\alpha}\int F^2(t)\,dt + \frac{i}{\hbar}\int\left(\frac{m\dot{x}^2}{2} - F(t)x\right)dt}$$

$$\int DF\, e^{-\frac{1}{2\alpha}\int F^2(t)\,dt - \frac{i}{\hbar}\int F(t)x\,dt} = Z_0\, e^{-\frac{\alpha}{2}\int x^2\,dt}$$

$$\Rightarrow \quad \langle G\rangle_F = \int Dx\, e^{i\int\left(\frac{m\dot{x}^2}{2} + \frac{i\alpha x^2}{2}\right)dt}$$

$$L_{eff} = \frac{m\dot{x}^2}{2} + \frac{i\alpha x^2}{2} = \frac{m\dot{x}^2}{2} - \frac{1}{2}m\omega^2x^2$$ where $\omega^2 = 0 - \frac{i\alpha}{m}$

Since the Lagrangian has exactly the same form as simple harmonic oscillator with frequency ω, we can write $\langle G \rangle$ as follows:

$$\langle G \rangle = \sqrt{\frac{m\omega}{2\pi i \sin[\omega(t_2-t_1)]}} \exp\left[\frac{m\omega\,(x_1^2+x_2^2)}{2\sin[\omega(t_2-t_1)]}\cos(\omega(t_2-t_1)) - 2x_1x_2\right]$$

The End.