» PHYS 501 «

Solutions to Homework Assignment

NO. 1

i) a dynamical system can be described by a functional, so called <u>action</u>, whose minimum gives the equations of motion. In other words, there exist a scalar function of system coordinates, and their time derivatives that can also have explicit time dependence, so called <u>Lagrangian</u>. If one integrate this function over time we obtain the action. The classical path of motion is the one which minimize this functional S.

$$S_{[2,1]} = \int_{t_0,\, Q(t_0)=Q_1}^{t,\, Q(t)=Q_2} dt'\, L(Q,\dot{Q},t) \qquad Q \equiv (q_1(t), q_2(t), \dots, q_{3N}(t))$$

$$\dot{Q} = \frac{dQ}{dt}$$

$$\delta S = \int_{t_0}^{t} dt' \left[\frac{\partial L}{\partial Q}.\delta Q + \frac{\partial L}{\partial \dot{Q}}.\delta \dot{Q} \right] \qquad \delta Q \Big|_{t'=t_0,t} = 0$$

$$\delta \dot{Q} = \dot{(\delta Q)} = \frac{d}{dt}\delta Q \implies \frac{\partial L}{\partial \dot{Q}}\delta \dot{Q} = \frac{\partial L}{\partial \dot{Q}} \frac{d}{dt}(\delta Q)$$

$$\frac{\partial L}{\partial \dot{Q}} \frac{d}{dt}(\delta Q) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\delta Q\right) - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}}\delta Q$$

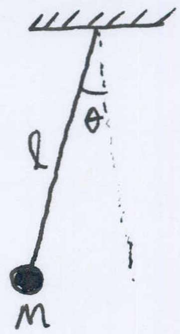
$$\implies \delta S = \int_{t_0}^{t} dt' \left[\left(\frac{\partial L}{\partial Q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}}\right).\delta Q \right] + \underbrace{\cancel{\frac{\partial L}{\partial \dot{Q}}\delta Q \Big|_{t_0}^{t}}}_{0}$$

$$\left(\frac{\partial L}{\partial Q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}}\right).\delta Q = \sum_{i=1}^{3N} \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right)\delta q^i$$

Since δq^i can be chosen arbitrarily independent of each other, $\delta S = 0$ implies that all coefficients are zero:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \qquad i = 1, \dots, 3N \quad (\text{Lagrange's eqs})$$

ii)



$$\mathcal{L} = T - V = \frac{1}{2} M v^2 - U_{gravity}$$

$$U_{gravity} = Mgl(1-\cos\theta)$$

$$v = l\dot{\theta} \quad \Rightarrow \quad \boxed{\mathcal{L} = \frac{1}{2} M l^2 \dot{\theta}^2 - Mgl(1-\cos\theta)}$$

$$\boxed{S = \int_{t_0,\ \theta_{t_0}=\theta_1}^{t,\ \theta_t=\theta_2} dt' \left[\frac{1}{2} M l^2 \dot{\theta}^2 - Mgl(1-\cos\theta)\right]}$$

$$H = p\dot{q} - \mathcal{L} \qquad p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = M l^2 \dot{\theta}$$

$$\Rightarrow H = M l^2 \dot{\theta}^2 - \frac{1}{2} M l^2 \dot{\theta}^2 + Mgl(1-\cos\theta)$$

$$H = \frac{1}{2} M l^2 \dot{\theta}^2 + Mgl(1-\cos\theta)$$

$$\Downarrow$$

$$\boxed{H = \frac{p_\theta^2}{2Ml^2} + Mgl(1-\cos\theta)}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

or

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta}, \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta}$$

$$\Rightarrow \quad M l^2 \ddot{\theta} = -Mgl\sin\theta$$

$$\Downarrow$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

for $\theta \ll 1 \Rightarrow \ddot{\theta} + \underbrace{\frac{g}{l}}_{\omega^2}\theta = 0 \Rightarrow \theta = A\cos\omega t + B\sin\omega t, \ \omega^2 = \frac{g}{l}$

construct the phase portrait, we obtain an equation for a surface of constant energy (since we know that the energy is conserved and the trajectories are lines of constant energy).

$$H = \frac{P_\theta^2}{2Ml^2} + Mgl(1-\cos\theta) \qquad \frac{dH}{dt} = 0$$

$$\Rightarrow \frac{P_\theta^2}{2Ml^2} + Mgl(1-\cos\theta) = \text{costant}$$

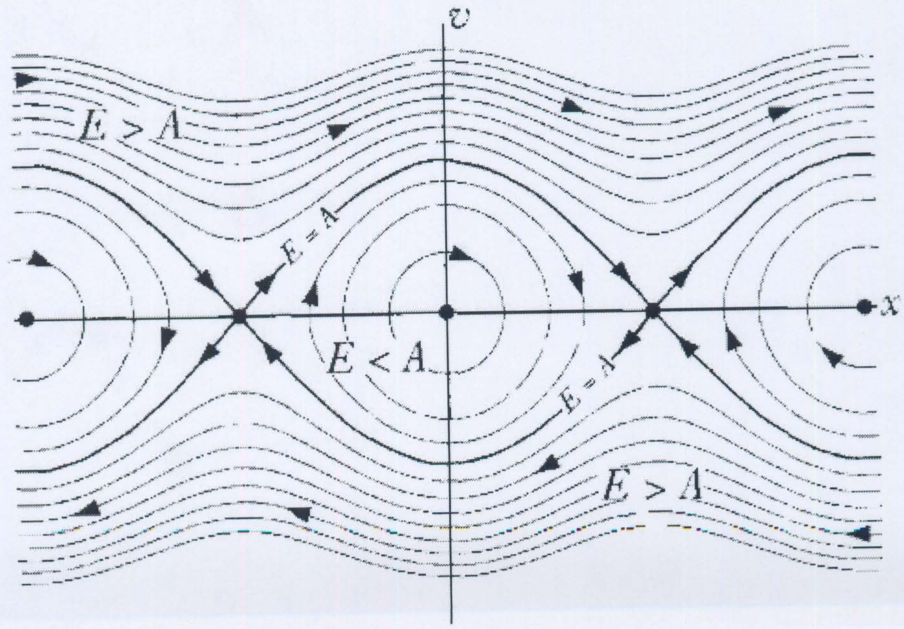
for $\theta \ll 1 \Rightarrow \frac{P_\theta^2}{2Ml^2} + \frac{Mgl\theta^2}{2} = cte \Rightarrow$ ellipse

for θ around π, $\cos\theta = \cos(\pi+\theta') = -\cos\theta' \simeq -1 + \frac{\theta'^2}{2} \Rightarrow \frac{P_\theta^2}{2Ml^2} - \frac{Mgl\theta'^2}{2} = cte$

$\Downarrow$

hyperbola

(b)



$V(x) = -A\cos x.$

2.

$$\rho = \frac{A}{2}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{A}{2}|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + (1-A)|\psi\rangle\langle\psi|$$

where $$|\psi\rangle = \frac{|\uparrow\uparrow\rangle + e^{i\theta}|\downarrow\downarrow\rangle}{\sqrt{2}}$$

$$\Rightarrow |\psi\rangle\langle\psi| = \frac{1}{2}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{2}e^{-i\theta}|\uparrow\uparrow\rangle\langle\downarrow\downarrow| + \frac{1}{2}e^{+i\theta}|\downarrow\downarrow\rangle\langle\uparrow\uparrow|$$
$$+ \frac{1}{2}|\downarrow\downarrow\rangle\langle\downarrow\downarrow|$$

$$\Rightarrow \rho = \frac{1}{2}|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{2}|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + \frac{e^{-i\theta}(1-A)}{2}|\uparrow\uparrow\rangle\langle\downarrow\downarrow| + \frac{e^{i\theta}(1-A)}{2}|\downarrow\downarrow\rangle\langle\uparrow\uparrow|$$

$$\Rightarrow \rho \doteq \frac{1}{2}\begin{array}{c} \begin{array}{cccc} |\uparrow\uparrow\rangle & |\uparrow\downarrow\rangle & |\downarrow\uparrow\rangle & |\downarrow\downarrow\rangle \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & (1-A)e^{-i\theta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1-A)e^{i\theta} & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\sigma_1^z\sigma_2^z = \sigma_1^z \otimes \sigma_2^z \doteq \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}$$

$$\langle \sigma_1 \sigma_2 \rangle = \mathrm{Tr}(\sigma_1 \otimes \sigma_2 \, \rho)$$

$$\sigma_1^z \otimes \sigma_2^z \rho \doteq \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & (1-A)e^{-i\theta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1-A)e^{i\theta} & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_1^z \otimes \sigma_2^z \rho \doteq \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & (1-A)e^{-i\theta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1-A)e^{i\theta} & 0 & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \mathrm{Tr}(\sigma_1^z \otimes \sigma_2^z \rho) = \frac{1}{2} + \frac{1}{2} = 1$$

$$\Longrightarrow \boxed{\langle \sigma_1^z \sigma_2^z \rangle = 1}$$

$$\langle \sigma_1^x \sigma_2^x \rangle = \mathrm{Tr}(\sigma_1^x \otimes \sigma_2^x \rho)$$

$$\sigma_1^x \otimes \sigma_2^x \rho \doteq \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & (1-A)e^{-i\theta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1-A)e^{i\theta} & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} (1-A)e^{+i\theta} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & (1-A)e^{-i\theta} \end{pmatrix}$$

$$\Longrightarrow \mathrm{Tr}\left(\sigma_1^x \otimes \sigma_2^x \rho\right) = \tfrac{1}{2}(1-A)\left(e^{i\theta} + e^{-i\theta}\right) = (1-A)\cos\theta$$

$$\Downarrow$$

$$\boxed{\langle \sigma_1^x \sigma_2^x \rangle = (1-A)\cos\theta}$$

$$\rho_1 = \mathrm{Tr}_2\, \rho = \tfrac{1}{2}|\uparrow\rangle\langle\uparrow| + \tfrac{1}{2}|\downarrow\rangle\langle\downarrow| \doteq \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\langle \sigma_1^z \rangle = \mathrm{Tr}\left(\sigma_1^z \rho\right) = \mathrm{Tr}\left(\frac{\sigma_1^z}{2}\right) = 0$$

$$\langle \sigma_1^x \rangle = \mathrm{Tr}\left(\sigma_1^x \rho\right) = \mathrm{Tr}\left(\frac{\sigma_1^x}{2}\right) = 0$$

The End.