» PHYS 501 «

Solutions

of

HW #4

Q1.

$$\vec{S} = \hbar\vec{\sigma} \quad , \quad |\vec{\sigma}| = 1/2$$

$$H = -\vec{\mu}\cdot\vec{B} \quad , \quad \vec{\mu} = g\mu_B\vec{S} = g\mu_B\hbar\vec{\sigma}$$

$$\vec{B} = \hat{z}B_0 + \hat{x}b_1 f(t)$$

$$\Rightarrow H = \underbrace{-g\mu_B\hbar B_0\sigma_z}_{\hat{H}_0} - \underbrace{b_1 g\mu_B\hbar f(t)\sigma_x}_{\hat{V}(t)}$$

$$C_n(t) = C_n(t\to-\infty) - \frac{i}{\hbar}\int_{-\infty}^{t} (V_I(t'))_{nm}\, C_m(t')\,dt'$$ $\to$ summation

first order :

$$C_n(\infty) = C_n(-\infty) - \frac{i}{\hbar}\int_{-\infty}^{\infty} (V_I(t'))_{nm}\, C_m(-\infty)\,dt'$$

two-level $\Rightarrow n=1,2$ $\quad \hat{z}\uparrow \to 1$, $\hat{z}\downarrow \to 2$

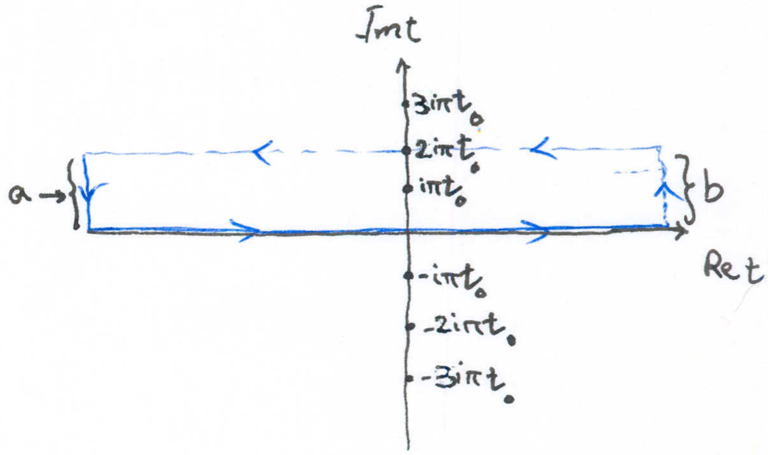
$$C_n(-\infty) = \delta_{n1}$$

$$\Rightarrow C_2(\infty) = \underbrace{\cancel{\delta_{21}}}_{0} - \frac{i}{\hbar}\int_{-\infty}^{\infty} (V_I(t'))_{21}\,dt'$$

$$(V_I(t'))_{21} = e^{ig\mu_B B_0 t'}\cdot(\underbrace{-b_1 g\mu_B\hbar}_{C} f(t')) = C\,e^{i\omega t'} f(t')$$

$$\Rightarrow \quad C_2(\infty) = -\frac{i}{\hbar} C \int_{-\infty}^{\infty} e^{i\omega t'} f(t')\, dt'$$

$$f(t') = \frac{e^{t'/t_0}}{1 + e^{t'/t_0}}$$



$$\oint f(z)\, dz = 2\pi i \, \mathrm{Res}_z(i\pi)$$

Since $f(z)$ on the a ~~or~~ b sides have nonzero value we should be careful when we integrate,

Note that $\quad f(t) = \dfrac{1}{1 + e^{-t/t_0}} = 1 - \dfrac{1}{1 + e^{t/t_0}} = 1 - f(-t)$

$$\Rightarrow f(t) + f(-t) = 1 \quad \xrightarrow{F.T} \quad f(\omega) + f^*(\omega) = 2\pi\, \delta(\omega)$$

$\Rightarrow \mathrm{Re} f(\omega) = \pi\, \delta(\omega) \Rightarrow$ This in infact comes from the a, b side integration

for the two other sides we have:

$$P\left[\int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt - e^{-2\pi\omega t_0} \int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt\right] = 2\pi i \cdot (-i e^{-\omega\pi t_0}) \cdot (-t_0)$$

excluding the Dirac delta function contribution

$$\Rightarrow \sinh \omega\pi t_0 \int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt = -\pi t_0$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt = \frac{i\pi t_0}{\sinh \omega\pi t_0} + \pi\delta(\omega)$$

$$\Rightarrow C_2(\infty) = -\frac{i}{\hbar} C \cdot \left[\frac{i\pi t_0}{\sinh[\omega\pi t_0]} + \pi\delta(\omega) \right]$$

$$\omega = g\mu_B |B_0| \quad , \quad C = -b_1 g\mu_B \hbar$$

$$\Rightarrow C_2(\omega) = \frac{b_1 g\mu_B \pi t_0}{\sinh[g\mu_B |B_0| \pi t_0]}$$

(ii) if $g\mu_B |B_0| t_0 \ll 1$ it means that $\omega t_0 \ll 1$

This means that for our system $f(t)$ behaves like an step function $\Theta(t)$ so we have

$$C_2(t) = -\frac{i}{\hbar} C_1 \int_{-\infty}^{t} \theta(t') e^{i\omega t'} dt' = -\frac{i}{\hbar} C_1 \left[\frac{e^{i\omega t} - 1}{i\omega} \right] \Theta(t)$$

$$\Longrightarrow \quad C_2(t) = \frac{b_1 g \mu_B}{g \mu_B |B_0|} \left[e^{i g \mu_B |B_0| t} - 1 \right] \Theta(t)$$

So the system remains into initial state for $t<0$ and from $t=0$ it starts t transition to 2nd state with the amplitude given above.

$$\mathcal{H}_0 = -K_0 \hat{S}_z^2 + E_0 S_x^2 \qquad K_0, E_0 > 0$$

(i) $E_0 = 0 \implies \mathcal{H} = -K_0 S_z^2$ $\qquad \hat{S}_z|M\rangle = \hbar M|M\rangle$

$M = -S, -S+1, \dots, S-1, S$

$\implies E_M = -K_0 \hbar^2 M^2$, so $|M\rangle$ & $|-M\rangle$ are degenerate.

(ii) $\frac{E_0}{K_0} \ll 1$, $S_x^2 = \left(\frac{S_+ + S_-}{2}\right)^2 = \frac{S_+^2 + S_-^2 + S_+S_- + S_-S_+}{4}$

$$\Delta_M^S = 2 \cdot \left\{ V^{-M,-(M-2)} \cdot \frac{1}{|E_M^0 - E_{-(M-2)}^0|} \cdot V^{-(M-2),-(M-4)} \cdot \frac{1}{|E_M^0 - E_{-(M-4)}^0|} \cdots\cdots V \cdot \frac{1}{|E_M^0 - E_{(M-2)}^0|} V^{M-2,M} \right\}$$

$$\Delta E_M(k) \equiv |E_M^0 - E_{-(M-2k)}^0| \qquad k = 1, \dots, M-1$$

$$V_M(k) = V^{-(M-2k+2),-(M-2k)} = \langle -(M-2k+2)|(+E_0\hat{S}_x^2)|-(M-2k)\rangle$$

$$\implies \Delta E_M(k) = +K_0\hbar^2[M^2 - (M-2k)^2] = +4K_0\hbar^2 k(M-k)$$

$$V_M(k) = E_0\hbar^2\left[(S+M-2k+2)(S+M-2k+1)(S-M+2k)(S-M+2k-1)\right]^{1/2}$$

$$\implies \Delta_M^S = 2 \cdot \left[\prod_{k=1}^{M-1} \frac{V_M(k)}{\Delta E_M(k)}\right] \cdot V^{-M,-(M-2)}$$

$$\Rightarrow \Delta_M = \frac{2 \cdot V}{(4K_0)^{M-1}} \cdot \left(\prod_{k=1} \frac{1}{k(M-k)} \right) \cdot \left(\frac{E_0}{4}\right) \cdot \left(\prod_{k=1} \tilde{V}_M(k) \right)$$

$$\downarrow$$

$$\frac{1}{[(M-1)!]^2}$$

where $\tilde{V}_m(k) = \frac{4V_m(k)}{E_0 t^2}$

$$V^{M-2,M} \prod_{k=1}^{M-1} \tilde{V}_m(k) = \prod_{k=1}^{M} \tilde{V}_m(k) = \frac{(S+M)!}{(S-M)!}$$

$$\Rightarrow \Delta_M^S = 2 \cdot \left(\frac{E_0}{16K_0}\right)^{M-1} \frac{\cdot E_0 \cdot}{4[(M-1)!]^2} \frac{(S+M)!}{(S-M)!}$$

$$\Rightarrow \Delta_M^S = A_M^S K_0 \left(\frac{E_0}{16K_0}\right)^M$$

$$A_M^S = \frac{8}{[(M-1)!]^2} \cdot \frac{(S+M)!}{(S-M)!}$$

Stirling's approx: $\lim_{n \to \infty} \left[\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \right] = 1$

$$A_S^S = \left[\frac{8}{((S-1)!)^2} \cdot (2S)! \right] = \frac{8 \cdot \sqrt{2\pi \times (2S)} \cdot \left(\frac{2S}{e}\right)^{2S}}{\left[\sqrt{2\pi(S-1)} \cdot \left(\frac{S-1}{e}\right)^{S-1}\right]^2}$$

$$\Rightarrow A_s^S = \frac{8\cdot\sqrt{4\pi S}\cdot 4^S\cdot S^{2S}\cdot e^{-2S}}{\left[\sqrt{2\pi S}\cdot S^{S-1}\cdot e^{-S+1}\right]^2} = \frac{2\cdot 4^{S+1}}{\sqrt{\pi S}}\cdot\left(\frac{S}{e}\right)^2$$

$$\Rightarrow \Delta_s^S = \frac{2\cdot 4^{S+1}\cdot K_0\cdot S^{3/2}}{\sqrt{\pi}\cdot e^2}\cdot\left(\frac{E_0}{16K_0}\right)^S$$

$$\ln\left[\frac{\Delta_s^S}{K_0}\right] = \ln C + \frac{3}{2}\ln S + S\ln\left(\frac{E_0}{4K_0}\right)$$

where $C = \frac{8}{\sqrt{\pi}}\cdot\frac{1}{e^2}$