1.

i) NORMAL MODES & NATURAL FREQUECIES

$$\mathcal{H} = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + g x_1 x_2 + \frac{1}{2} m_1 \omega_1^2 x_1^2 + \frac{1}{2} m_2 \omega_2^2 x_2^2$$

By $\boxed{p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m_i \dot{x}_i}$ and $\boxed{\dot{p}_i = -\frac{\partial H}{\partial x_i}}$;

$$\begin{cases} \ddot{x}_1 = -\frac{g}{m_1} x_2 - \omega_1^2 x_1 \\ \ddot{x}_2 = -\frac{g}{m_2} x_1 - \omega_2^2 x_2 \end{cases}$$

In matrix language

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \omega_1^2 & \frac{g}{m_1} \\ \frac{g}{m_2} & \omega_2^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

① Diagonalize $\begin{pmatrix} \omega_1^2 & g/m_1 \\ g/m_2 & \omega_2^2 \end{pmatrix}$

$$\begin{vmatrix} \omega_1^2 - \lambda & g/m_1 \\ g/m_2 & \omega_2^2 - \lambda \end{vmatrix} = 0$$

$$(\omega_1^2 - \lambda)(\omega_2^2 - \lambda) - g^2/m_1 m_2 = 0$$

$$\lambda^2 - (\omega_1^2+\omega_2^2)\lambda + \omega_1^2\omega_2^2 - g^2/m_1m_2 = 0$$

$$\therefore \lambda_\pm = \frac{1}{2}\left[(\omega_1^2+\omega_2^2) \pm \sqrt{(\omega_1^2+\omega_2^2)^2 - 4\omega_1^2\omega_2^2 + 4g^2/m_1m_2} \right]$$

$$= \frac{1}{2}\left[(\omega_1^2+\omega_2^2) \pm \sqrt{(\omega_1^2-\omega_2^2)^2 + 4g^2/m_1m_2} \right]$$

$$\equiv \Omega_\pm^2$$

i.e.

$$P^{-1}\begin{pmatrix} \omega_1^2 & g/m_1 \\ g/m_2 & \omega_2^2 \end{pmatrix} P = \begin{pmatrix} \Omega_+^2 & 0 \\ 0 & \Omega_-^2 \end{pmatrix}$$

$$P = \begin{pmatrix} \mathbf{e}_+ & \mathbf{e}_- \end{pmatrix}$$ where $\mathbf{e}_+, \mathbf{e}_-$ are eigenvectors

② Find eigenvectors

a) λ_+

$$\begin{pmatrix} \omega_1^2-\lambda_+ & g/m_1 \\ g/m_2 & \omega_2^2-\lambda_+ \end{pmatrix}\begin{pmatrix} \alpha_+ \\ \beta_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\alpha_+ = 1 \; ; \; \beta_+ = -\frac{m_1}{g}(\omega_1^2-\lambda_+)$$

$$= -\frac{m_1}{2g}\left[2\omega_1^2-\omega_1^2-\omega_2^2-\sqrt{(\omega_1^2-\omega_2^2)^2+4g^2/m_1m_2}\right]$$

$$= -\frac{m_1}{2g}\left[\omega_1^2-\omega_2^2-\sqrt{(\omega_1^2-\omega_2^2)^2+4g^2/m_1m_2}\right]$$

b) λ_-

$$\begin{pmatrix} \omega_1^2-\lambda_- & g/m_1 \\ g/m_2 & \omega_2^2-\lambda_- \end{pmatrix}\begin{pmatrix} \alpha_- \\ \beta_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\alpha_- = 1 \; ; \; \beta_- = -\frac{m_1}{g}(\omega_1^2-\lambda_-)$$

$$= -\frac{m_1}{2g}\left[(\omega_1^2-\omega_2^2)+\sqrt{(\omega_1^2-\omega_2^2)^2+4g^2/m_1m_2}\right]$$

$$\therefore\ P = \begin{pmatrix} 1 & 1 \\ \beta_+ & \beta_- \end{pmatrix},\quad P^{-1} = \frac{1}{\beta_- - \beta_+}\begin{pmatrix} \beta_- & -1 \\ -\beta_+ & 1 \end{pmatrix}$$

③ using $\boxed{① \& ②}$, rewrite the equation

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -P\begin{pmatrix} \Omega_+^2 & 0 \\ 0 & \Omega_-^2 \end{pmatrix} P^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\frac{d^2}{dt^2} P^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \Omega_+^2 & 0 \\ 0 & \Omega_-^2 \end{pmatrix} P^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

let $P^{-1}x = q$, then

$$\frac{d^2}{dt^2}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = -\begin{pmatrix} \Omega_+^2 & 0 \\ 0 & \Omega_-^2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

and (q_1, q_2) are normal modes!

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{\beta_- - \beta_+} \begin{pmatrix} \beta_- & -1 \\ -\beta_+ & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \frac{1}{\beta_- - \beta_+} \begin{pmatrix} \beta_- x_1 - x_2 \\ -\beta_+ x_1 + x_2 \end{pmatrix}$$

$\boxed{\text{NOTE}}$

$$\beta_- - \beta_+ = \frac{m_1}{g} \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4g^2/m_1 m_2}$$

$$\beta_+ = \frac{m_1}{2g} \left[(\omega_1^2 - \omega_2^2) - \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4g^2/m_1 m_2} \right]$$

$$\beta_- = \frac{m_1}{2g} \left[(\omega_1^2 - \omega_2^2) + \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4g^2/m_1 m_2} \right]$$

ii) HAMILTON – JACOBI EQUATION

(a) $$\frac{\partial S}{\partial Q} = \int dt \, \frac{\partial \mathcal{L}}{\partial Q} \qquad // \; S = \int dt \, \mathcal{L}.$$

$$= \int dt \, \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{Q}}\right) \qquad // \text{ E-L eq.}$$

$$= \frac{\partial \mathcal{L}}{\partial \dot{Q}} \qquad // \text{ integration}$$

$$= P$$

(b) $$S = \int P dQ - H dt$$

$$\frac{\partial S}{\partial t} = -H(P, Q; t)$$

(a) + (b) : $$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial Q}, Q; t\right) = 0$$

FOR $V(r)$ CASE IN 2D, H-J eq. is :

$$\frac{\partial S}{\partial t} + \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + V(r) = 0$$

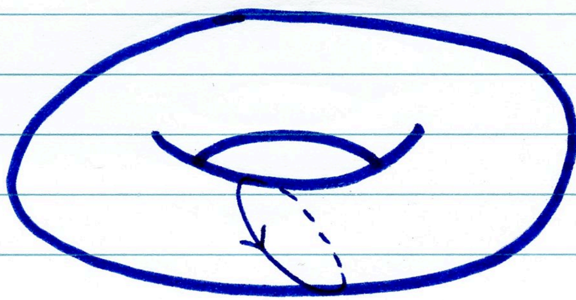
$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x_1}\right)^2 + \frac{1}{2m}\left(\frac{\partial S}{\partial x_2}\right)^2 + V(r) = 0$$

2D HARMONIC OSCILLATOR, in terms of (x_1, x_2) and (p_1, p_2)

$$\mathcal{H} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2}m\omega^2 x_2^2$$

* For constant action,
 (p_1, p_2) and (x_1, x_2) make two independent circles $\mathbb{C}_1, \mathbb{C}_2$. AND these circles $\mathbb{C}_1 \times \mathbb{C}_2$ make 2-torus.

* In cartesian coordinates, the orbit closes because the two periods are same.



#2. $|\psi_a\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle)$

$|\psi_b\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$

$|\psi\rangle = \frac{1}{\sqrt{3}}(|\psi_a\rangle + |\psi_b\rangle)$

i) DENSITY MATRIX

$$\hat{\rho} = \frac{1-A^2}{2}|\psi_a\rangle\langle\psi_a| + \frac{1-A^2}{2}|\psi_b\rangle\langle\psi_b| + P|\psi\rangle\langle\psi|, \quad P=A^2$$

Basis : $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$

① $\langle b'|\psi_a\rangle\langle\psi_a|b''\rangle$

$$\begin{array}{c c} & \begin{array}{cccc} \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \end{array} \\ \begin{array}{c} \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{array} & \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

② $\langle b'|\psi_b\rangle\langle\psi_b|b''\rangle$

$$\begin{array}{c c} & \begin{array}{cccc} \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \end{array} \\ \begin{array}{c} \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{array} & \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix} \end{array}$$

③ $\langle b' | \psi \rangle \langle \psi | b'' \rangle$

$$
\begin{array}{c|cccc}
 & \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \\
\hline
\uparrow\uparrow & 4/6 & 2/6 & 0 & 2/6 \\
\uparrow\downarrow & 2/6 & 1/6 & 0 & 1/6 \\
\downarrow\uparrow & 0 & 0 & 0 & 0 \\
\downarrow\downarrow & 2/6 & 1/6 & 0 & 1/6
\end{array}
$$

$$\therefore \quad \rho = \frac{1-A^2}{2}\left(① + ② \right) + p\left(③ \right)$$

$$= \frac{1-p}{2}\begin{pmatrix} 1 & 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix} + \frac{p}{6}\begin{pmatrix} 4 & 2 & 0 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 3+p & \frac{3}{2}+\frac{p}{2} & 0 & \frac{3}{2}+\frac{p}{2} \\ \frac{3}{2}+\frac{p}{2} & \frac{3}{2}-\frac{p}{2} & 0 & p \\ 0 & 0 & 0 & 0 \\ \frac{3}{2}+\frac{p}{2} & p & 0 & \frac{3}{2}-\frac{p}{2} \end{pmatrix}$$

EXPECTATION VALUES

$$\sigma_1^z \sigma_2^z = \begin{array}{c} \\ \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{array} \begin{array}{c} \begin{array}{cccc} \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\langle \hat{\sigma}_1^z \hat{\sigma}_2^z \rangle = \mathrm{Tr}\left[\rho\left(\sigma_1^z \sigma_2^z\right)\right]$$

$$= \frac{1}{6}\mathrm{Tr}\left[\begin{pmatrix} 3+p & 3/2+p/2 & 0 & 3/2+p/2 \\ 3/2+p/2 & 3/2-p/2 & 0 & p \\ 0 & 0 & 0 & 0 \\ 3/2+p/2 & p & 0 & 3/2-p/2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{6}\left[3+p-3/2+p/2+3/2-p/2\right]$$

$$= \frac{1}{6}(3+p)$$

$$\sigma_1^z \sigma_2^x = \begin{array}{c} \\ \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{array} \begin{array}{c} \begin{array}{cccc} \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$\langle \hat{\sigma}_1^z \hat{\sigma}_2^x \rangle = \mathrm{Tr}\left[\rho\left(\sigma_1^z \sigma_2^x\right)\right]$$

$$= \frac{1}{6}\left[3/2+p/2+3/2+p/2+0+0\right]$$

$$= \frac{1}{6}(3+p)$$

ii) REDUCED MATRIX

$$\rho = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$ where A, B, C, D are 2X2-matrix.

then, $$\bar{\rho} = \begin{pmatrix} TrA & TrB \\ TrC & TrD \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} \frac{9}{2} + \frac{p}{2} & p \\ p & \frac{3}{2} - \frac{p}{2} \end{pmatrix}$$

EXPECTATION VALUES

$$\sigma_1^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle \hat{\sigma}_1^z \rangle = Tr[\bar{\rho}\, \sigma_1^z]$$

$$= \frac{1}{6} \left(\frac{9}{2} + \frac{p}{2} - \frac{3}{2} + \frac{p}{2} \right)$$

$$= \frac{1}{6} (3 + p)$$

$$\sigma_1^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\langle \hat{\sigma}_1^x \rangle = Tr[\bar{\rho}\, \sigma_1^x]$$

$$= \frac{1}{6} (p + p)$$

$$= p/3$$

#3.

i) $G_{00}(t,0|F(t))$

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 - \omega_0^2 x^2) - F(t)x$$

$$\langle 0|\hat{G}(t,0|F(t))|0\rangle$$

$$= \int dx_1 dx_2 \langle 0|x_2\rangle\langle x_1|0\rangle \langle x_2|\hat{G}(t,0|F(t))|x_1\rangle$$

$$= \int dx_1 dx_2 \langle 0|x_2\rangle\langle x_1|0\rangle \int_{q(0)=x_1}^{q(t)=x_2} \mathcal{D}[q(t)]\, e^{\frac{i}{\hbar}S}$$

let $S_0 = \int dt\, \mathcal{L}_0 = \int dt \frac{m}{2}(\dot{x}^2 - \omega_0^2 x^2)$

then,

$$= \int dx_1 dx_2 \langle 0|x_2\rangle\langle x_1|0\rangle \int_{q(0)=x_1}^{q(t)=x_2} \mathcal{D}[q(t)]\, e^{\frac{i}{\hbar}S_0}\, e^{-\frac{i}{\hbar}\int d\tau\, F(\tau)q}$$

$$= \int dx_1 dx_2 \langle 0|x_2\rangle\langle x_1|0\rangle \int_{q(0)=x_1}^{q(t)=x_2} \mathcal{D}[q(t)]\, e^{\frac{i}{\hbar}S_0}\left[1 - \frac{i}{\hbar}\int d\tau\, F(\tau)q + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2\int d\tau d\tau' \ldots\right]$$

① $\int dx_1 dx_2 \ \langle 0|x_2\rangle\langle x_1|0\rangle \int \mathcal{D}[q(\tau)]\, e^{\frac{i}{\hbar}S_0}$

$= \langle 0|\hat{G}(t)|0\rangle = 1.$

② $\int dx_1 dx_2 \ \langle 0|x_2\rangle\langle x_1|0\rangle \int \mathcal{D}[q(\tau)]\, e^{\frac{i}{\hbar}S_0} \left\{ -\frac{i}{\hbar}\int d\tau\, F(\tau)\, q(\tau) \right\}$

$= \langle 0| \left\{ -\frac{i}{\hbar}\int d\tau\, F(\tau)\, q(\tau) \right\} |0\rangle.$

$= -\frac{i}{\hbar}\int d\tau \ F(\tau)\ \langle 0| e^{\frac{i}{\hbar}H_0\tau}\, q\, e^{-\frac{i}{\hbar}H_0\tau} |0\rangle$

$= -\frac{i}{\hbar}\int d\tau\, F(\tau)\ \langle 0|q|0\rangle$

$\propto \quad \langle 0|a+a^{+}|0\rangle = 0$

③ $\int dx_1 dx_2 \, \langle 0|x_2\rangle\langle x_1|0\rangle \int \mathcal{D}[q(\tau)] \, e^{\frac{i}{\hbar}S_0} \left\{ \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 \int d\tau d\tau' \, F(\tau)F(\tau')q(\tau)q(\tau') \right\}$

$$= -\frac{1}{\hbar^2} \underbrace{\frac{1}{2}\int_{t'}^{t} d\tau \int_0^{t'} d\tau'}_{\circledast \int_0^t d\tau \int_0^\tau d\tau'} \langle 0| F(\tau)F(\tau')q(\tau)q(\tau') |0\rangle$$

$$= -\frac{1}{\hbar^2}\int_0^t d\tau \int_0^\tau d\tau' \, F(\tau)F(\tau') \langle 0| e^{\frac{i}{\hbar}H_0\tau} q \, e^{\frac{i}{\hbar}H_0(\tau'-\tau)} q \, e^{-\frac{i}{\hbar}H_0\tau'} |0\rangle$$

$$= -\frac{1}{\hbar^2}\int_0^t d\tau \int_0^\tau d\tau' \, F(\tau)F(\tau') \, e^{i\omega_0(\tau-\tau')} \langle 0|q^2|0\rangle$$

$\boxed{\text{NOTE}}$ $q = \sqrt{\frac{\hbar}{2m\omega_0}}\,(a+a^\dagger)$

$$= -\frac{1}{\hbar^2}\cdot\frac{\hbar}{2m\omega_0}\int_0^t d\tau \int_0^\tau d\tau' \, F(\tau)F(\tau') \, e^{i\omega_0(\tau-\tau')} \underset{\substack{\parallel \\ 1.}}{\langle 0|(a+\cancel{a^\dagger})(\cancel{a}+a^\dagger)|0\rangle}$$

zero zero

$$= -\frac{1}{2m\hbar\omega_0}\int_0^t d\tau \int_0^\tau d\tau' \, F(\tau)F(\tau') \, e^{i\omega_0(\tau-\tau')}$$

① + ② + ③ ...

$$= 1 - \frac{1}{2m\hbar\omega_0}\int_0^t d\tau \int_0^\tau d\tau' \, F(\tau)\, F(\tau')\, e^{i\omega_0(\tau-\tau')} + \cdots$$

$$= \exp\left\{ -\frac{1}{2m\hbar\omega_0}\int_0^t d\tau \int_0^\tau d\tau' \, F(\tau)\, F(\tau')\, e^{i\omega_0(\tau-\tau')} \right\}$$

ii) SIMILARLY.

① $\int dx_1 dx_2 \langle 1|x_2\rangle\langle x_1|0\rangle \int \mathcal{D}[q(\tau)]\, e^{\frac{i}{\hbar}S_0}$

$= \langle 1|\hat{G}(t)|0\rangle = 0$

$\because \langle 0|\hat{G}(t)|0\rangle = 1$

② $\int dx_1 dx_2 \langle 1|x_2\rangle\langle x_1|0\rangle \int \mathcal{D}[q(t)]\, e^{\frac{i}{\hbar}S_0} \left\{ -\frac{i}{\hbar}\int d\tau\, F(\tau) q(\tau) \right\}$

$= -\frac{i}{\hbar}\int d\tau\, \langle 1| F(\tau) q(\tau) |0\rangle$

$= -\frac{i}{\hbar}\int d\tau\, F(\tau)\, \langle 1| e^{\frac{i}{\hbar}H_0\tau}\, q\, e^{-\frac{i}{\hbar}H_0\tau} |0\rangle$

zero.

$= -\frac{i}{\hbar}\int d\tau\, F(\tau)\, e^{i\omega_0\tau} \langle 1| \sqrt{\frac{\hbar}{2m\omega_0}}\, (\cancel{a} + a^{\dagger}) |0\rangle$

$= -\frac{i}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}} \int d\tau\, F(\tau)\, e^{i\omega_0\tau}$

③ $\int dx_1 dx_2 \langle 1|x_2\rangle\langle x_1|0\rangle \int \mathcal{D}[q(t)]\, e^{\frac{i}{\hbar}S_0} \left\{ \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 \int d\tau d\tau'\, F(\tau)F(\tau')\, q(\tau) q(\tau') \right\}$

$\propto \langle 1|\, q^2 \,|0\rangle$

$\propto \langle 1|\, (a+a^+)(\cancel{a}+a^+)|0\rangle$ zero

$= \langle 1|\, aa^+ + a^+a^+ \,|0\rangle$

$= 0$

① + ② + ③ ...

$= 0 + \left(-\frac{i}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}} \int d\tau\, F(\tau)\, e^{i\omega_0 \tau}\right) + 0 \ \cdots$

$= -\frac{i}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}} \int d\tau\, F(\tau)\, e^{i\omega_0 \tau} \left(1 + \left(-\frac{1}{2m\omega_0\hbar}\int_0^t d\tau \int_0^\tau d\tau'\, F(\tau)F(\tau')\, e^{i\omega_0(\tau-\tau')}\right)\right)$

$= -\frac{i}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}} \int d\tau\, F(\tau)\, e^{i\omega_0 \tau} \cdot G_{00}(t, 0\,|\,F(t))$