#1. (i)

$$\langle x_N, t_N | x_1, t_1 \rangle = \int dx_{N-1} \, \langle x_N, t_N | x_{N-1}, t_{N-1} \rangle \langle x_{N-1}, t_{N-1} | x_1, t_1 \rangle$$

$$= \int dx_{N-1} \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \exp\left[\left(\frac{im}{2\hbar}\right) \frac{(x_N - x_{N-1})^2}{\Delta t} + \frac{i}{\hbar} q A (x_N - x_{N-1}) - \frac{i}{\hbar} q V \Delta t \right] \langle x_{N-1}, t_{N-1} | x_1, t_1 \rangle$$

(x_N, t_N) : final state

$x_N - x_{N-1} \equiv \xi$, $t_N - t_{N-1} \equiv \Delta t$

then,

$$\langle x, t+\Delta t | x_0, t_0 \rangle = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int d\xi \exp\left(\frac{im\xi^2}{2\hbar\Delta t}\right) \exp\left(\frac{i}{\hbar} q A \xi\right) \exp\left(-\frac{i}{\hbar} q V \Delta t\right)$$

$$\times \langle x - \xi, t | x, t \rangle$$

since ξ, Δt are small,

$$= \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int d\xi \exp\left(\frac{im\xi^2}{2\hbar\Delta t}\right) \left[1 + \frac{i}{\hbar} q A \xi + \left(\frac{i}{\hbar}\right)^2 q^2 A^2 \xi^2 \right] \left[1 - \frac{i}{\hbar} q V \Delta t \right]$$

$$\times \langle x - \xi, t | x, t \rangle.$$

$$\langle x, t | x, t \rangle + \Delta t \frac{\partial}{\partial t} \langle x, t | x, t \rangle = [\ \dots\ ] \left[\langle x, t | x, t \rangle - \xi \frac{\partial}{\partial x} \langle\ \rangle + \frac{\xi^2}{2} \frac{\partial^2}{\partial x^2} \langle\ \rangle \right].$$

$$\sum_\alpha \psi_\alpha^*(x,t) \left[1 + \Delta t \frac{\partial}{\partial t} \right] \psi_\alpha(x,t)$$

$$= \sum_\alpha \psi_\alpha^*(x,t) [\ \dots\ ] \left[1 - \xi \frac{\partial}{\partial x} + \frac{\xi^2}{2} \frac{\partial^2}{\partial x^2} \right] \psi_\alpha(x,t).$$

Compare left-hand side and right-hand side, order Δt

$$(\text{l.h.s}) = \Delta t \frac{\partial}{\partial t}\psi(x,t)$$

NOT acting on A.

$$(\text{r.h.s}) = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\int_{-\infty}^{\infty} d\xi\, e^{\frac{im\xi^2}{2\hbar\Delta t}}\left[\left\{\frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{i}{\hbar}\frac{\partial}{\partial x}qA + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 q^2 A^2\right\}\xi^2 - \frac{i}{\hbar}qV\Delta t\right]\psi(x,t)$$

$\boxed{\text{NOTE}}$

$$\int d\xi\, e^{\frac{im\xi^2}{2\hbar\Delta t}} = \sqrt{\frac{2\pi i\hbar\Delta t}{m}}$$

$$\int d\xi\, \xi e^{\frac{im\xi^2}{2\hbar\Delta t}} = 0$$

$$\int d\xi\, \xi^2 e^{\frac{im\xi^2}{2\hbar\Delta t}} = \sqrt{\frac{2\pi i\hbar\Delta t}{m}}\left(\frac{i\hbar\Delta t}{m}\right)$$

$$(\text{r.h.s}) = \left[\frac{1}{2}\left(\frac{i\hbar\Delta t}{m}\right)\left(\frac{\partial^2}{\partial x^2} - 2\left(\frac{i}{\hbar}\frac{\partial}{\partial x}\right)(qA) + \left(\frac{i}{\hbar}\right)^2 q^2A^2\right) - \frac{i}{\hbar}V\Delta t\right]\psi(x,t)$$

$$= \left[\frac{i\hbar}{2m}\left(\frac{\partial}{\partial x} - \frac{i}{\hbar}qA\right)^2 - \frac{i}{\hbar}qV\right]\Delta t\, \psi(x,t)$$

$$= \left[\frac{i\hbar}{2m}\frac{i^2}{\hbar^2}\left(-i\hbar\frac{\partial}{\partial x} - qA\right)^2 - \frac{i}{\hbar}qV\right]\Delta t\, \psi(x,t)$$

$$\therefore\ i\hbar\partial_t\psi = \left[\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} - qA\right)^2 + qV\right]\psi(x,t)$$

$$\Downarrow \text{3D}$$

$$i\hbar\partial_t\psi = \left[\frac{1}{2m}\left(-i\hbar\vec{\nabla} - q\vec{A}\right)^2 + qV\right]\psi(x,t)$$

(ii)

$$A = \frac{B_0}{2}(-y, x, 0) \quad , \quad V = 0$$

$$\dot{z}_{cl}(0) = 0 \quad ; \quad z_{cl}(\tau) = z_{cl}(0)$$

① $$S_{cl}(2,1) = \int_0^\tau dt \; \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB}{2}(-\dot{x}y + y\dot{x})$$

E.O.M

$$\ddot{x} - \frac{qB_0}{2m}\dot{y} = 0$$

$$\ddot{y} + \frac{qB_0}{2m}\dot{x} = 0$$

$$\ddot{z} = 0$$

i.e. $$x(t) = A\cos(\omega t/2) + B\sin(\omega t/2) + C$$
$$y(t) = B\cos(\omega t/2) - A\sin(\omega t/2) + D$$

with $x(\tau) = x_2$, $x(0) = x_1$
$y(\tau) = y_2$, $y(0) = y_1$
$\omega = qB_0/m$

$$A = -\frac{1}{2}\left((x_2 - x_1) + \frac{\sin\omega\tau/2}{1-\cos\omega\tau/2}(y_2 - y_1)\right)$$

$$B = \frac{1}{2}\left(-(y_2 - y_1) + \frac{\sin\omega\tau/2}{1-\cos\omega\tau/2}(x_2 - x_1)\right)$$

$$C = \frac{1}{2}\left((x_2 + x_1) + \frac{\sin\omega\tau/2}{(1-\cos\omega\tau/2)}(y_2 - y_1)\right)$$

$$D = \frac{1}{2}\left((y_2 + y_1) - \frac{\sin\omega\tau/2}{(1-\cos\omega\tau/2)}(x_2 - x_1)\right).$$

Now, go back to the action

$$\int_0^\tau dt \; \frac{m}{2}\left[\dot{x}^2+\dot{y}^2-\omega \dot{x} y+\omega y \dot{x}\right]$$

$$= \frac{m}{2}\cdot \omega^2 \int_0^\tau dt \; (A^2+B^2)-(A^2+B^2)-(BD+AC)\cos \omega t/2+(AD-BC)\sin \omega t/2$$

$$BC = \frac{1}{4}\Big[\frac{\sin}{1-\cos}(x_2-x_1)(x_2+x_1)-\frac{\sin}{1-\cos}(y_2-y_1)^2-(y_2-y_1)(x_2+x_1)$$

$$+\left(\frac{\sin}{1-\cos}\right)^2 (x_2-x_1)(y_2-x_1)\Big]$$

$$AD = -\frac{1}{4}\Big[\frac{\sin}{1-\cos}(y_2-y_1)(y_2+y_1)-\frac{\sin}{1-\cos}(x_2-x_1)^2+(x_2-x_1)(y_2+y_1)$$

$$-\left(\frac{\sin}{1-\cos}\right)^2 (x_2-x_1)(y_2-x_1)\Big)$$

$$AD-BC = -\frac{1}{4}\Big[\frac{\sin}{1-\cos}\Big\{ (y_2-y_1)(y_2+y_1)-(x_2-x_1)^2-(y_2-y_1)^2$$

$$+(x_2-x_1)(x_2+x_1)\Big\}-(y_2-y_1)(x_2+x_1)$$

$$+(x_2-x_1)(y_2+y_1)\Big]$$

$$AC+BD = \frac{1}{4}\Big[-\Big(\frac{\sin}{1-\cos}\Big)^2\big((x_2-x_1)^2+(y_2-y_1)^2\big)$$

$$+\Big(\frac{\sin}{1-\cos}\Big)\Big\{(x_2-x_1)(y_2+y_1)-(y_2-y_1)(x_2+x_1)\Big\}$$

$$-(y_2-y_1)(y_2+y_1)-(x_2-x_1)(x_2+x_1)\Big].$$

$$-(BD+AC)(\cos\omega\tau/2-1)+(AD-BC)\sin\omega\tau/2.$$

$$\therefore\ S_{cl}(2,1) = \frac{m\omega^2}{4}\cot\omega\tau/2\Big[\,|\vec{r}_2-\vec{r}_1|^2 - \hat{z}\cdot(\vec{r}_2\times\vec{r}_1)\Big].$$

② $A(\tau) = \int_{x(0)=0}^{x(\tau)=0} \mathcal{D}[x(t)]\; e^{\frac{i}{\hbar}\int_0^\tau dt\; \frac{1}{2}m\dot{x}^2 + q\dot{x}A}$.

$A_z(\tau) = \sqrt{\dfrac{m}{2\pi i\hbar\tau}}$ (free particle)

$A_{x,y}(\tau) = \int_{\vec{x}(0)=0}^{\vec{x}(\tau)=0} \mathcal{D}[\vec{x}(t)]\; e^{\frac{i}{\hbar}\int_0^\tau dt\; \frac{1}{2}m\dot{\vec{x}}^2 + q\dot{\vec{x}}\vec{A}}$, $\vec{x} = (x, y)$ 2-dim.

0 , Boundary Condition

$$\frac{m}{2}\dot{\vec{x}}^2 + q\dot{\vec{x}}\vec{A} = \frac{m}{2}\left[\cancelto{0}{\frac{d}{dt}\vec{x}\cdot\dot{\vec{x}}} - \vec{x}\ddot{\vec{x}}\right] + \frac{qB_0}{2}\left[(\dot{x},\dot{y})\cdot(-y,x)\right]$$

$$\cancel{= \frac{m}{2}\left[-\vec{x}\cdot\frac{\partial^2}{\partial t^2}\vec{x}\right.}$$

$$= \frac{m}{2}\left[-(x\ \ y)\begin{pmatrix} \partial_t^2 & 0 \\ 0 & \partial_t^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (x\ \ y)\begin{pmatrix} 0 & \omega\partial_t \\ -\omega\partial_t & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right]$$

$$= \frac{m}{2}(x\ \ y)\underbrace{\begin{pmatrix} -\partial_t^2 & \omega\partial_t \\ -\omega\partial_t & -\partial_t^2 \end{pmatrix}}_{\equiv D}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$A_{\vec{x}}(\tau) = \left(\frac{m}{2\pi i\hbar\tau}\right)\frac{1}{\det D}$$

$$= \left(\frac{1}{2\pi i\hbar\tau}\right)\prod_{n=1}^{\infty}\lambda_n$$, λ_n are eigenvalues of D.

$$\begin{pmatrix} -\partial_t^2-\lambda_n & \omega\partial_t \\ -\omega\partial_t & -\partial_t^2-\lambda_n \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\lambda_n^+ = \partial_t - W$$

$$= \frac{2n\pi}{\tau} - W$$

Because of boundary conditions $\vec{x}(\tau)=\vec{x}(0)$

$$\lambda_n^- = \frac{2n\pi}{\tau} + W$$

$$\lambda_n \equiv \lambda_n^+ \lambda_n^-$$

$$= \left(\frac{2n\pi}{\tau}\right)^2 - W^2$$

$$\rightarrow 1 - \left(\frac{W\tau}{2n\pi}\right)^2$$

$$\prod_n \lambda_n = \prod_{n=1}^{\infty}\left[1-\left(\frac{W\tau}{2n\pi}\right)^2\right]$$

NOTE $\quad \frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left[1-\frac{z^2}{n^2}\right]$

$$\therefore \prod_n \lambda_n = \frac{W\tau/2}{\sin W\tau/2}$$

$$A(\tau) = \left(\frac{m}{2\pi i\hbar\tau}\right)^{3/2} \frac{W\tau}{2\sin W\tau/2}$$

#2. (i)

$$f_k(\Omega) = -\frac{2m}{4\pi\hbar^2}\int d^dx\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}'}\, V(\vec{x}')$$

$$\frac{d\sigma}{d\Omega} = |f_k(\Omega)|^2$$

$$\sigma = \int |f_k(\Omega)|^2\, d\Omega$$

① $d = 1$, $V(r) = V_0 e^{-\kappa r}$

$$f_k = -\frac{2m}{4\pi\hbar^2}\int_0^\infty dr\, e^{i(k-k')r}\, V_0 e^{-\kappa r}$$

$$= -\frac{2mV_0}{4\pi\hbar^2}\int_0^\infty dr\, e^{-(\kappa - i(k-k'))r}$$

$$= \frac{+2mV_0}{4\pi\hbar^2}\left.\frac{e^{-\{\kappa - i(k-k')\}}}{\kappa - i(k-k')}\right|_0^\infty$$

$$= -\frac{2mV_0}{4\pi\hbar^2}\cdot\frac{1}{\kappa - i(k-k')}$$

$$|f_k|^2 = \left|\frac{2mV_0}{4\pi\hbar^2}\right|^2\cdot\frac{2\kappa.}{\kappa^2+(k-k')^2} = \sigma$$ No Angle Dependence

② $d=2$, $V(r) = \frac{V_0}{\sqrt{r}} e^{-\kappa r}$

$$f_k = -\frac{2m}{4\pi\hbar^2}\int d^2r' e^{i(\vec{k}-\vec{k}')\cdot\vec{r}'} \frac{V_0}{\sqrt{r'}} e^{-\kappa r'}$$

$$= -\frac{2mV_0}{4\pi\hbar^2}\int d^2r' \, e^{i|\vec{k}-\vec{k}'|r'\cos\theta'} \frac{e^{-\kappa r'}}{\sqrt{r'}}$$

$$= -\frac{2mV_0}{4\pi\hbar^2}\int_0^\infty r'dr' \int_0^{2\pi} d\theta' \, e^{i|\vec{k}-\vec{k}'|r'\cos\theta'} \frac{e^{-\kappa r'}}{\sqrt{r'}}$$

$$= -\frac{2mV_0}{4\pi\hbar^2} F(|\vec{k}-\vec{k}'|, \kappa)$$

$$|f_k|^2 = \left|\frac{2mV_0}{4\pi\hbar^2}\right|^2 |F|^2$$

$$\sigma = \left|\frac{2mV_0}{4\pi\hbar^2}\right|^2 \int_0^{2\pi} d\theta \, |F|^2$$

③ $d=3$, $V = \frac{V_0}{r} e^{-\kappa r}$

$$f_k(\Omega) = -\frac{2m}{4\pi\hbar^2} \int d^3r' \, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}'} \frac{V_0}{r'} e^{-\kappa r'}$$

$$= -\frac{2mV_0}{4\pi\hbar^2} \cdot 2\pi \int d\cos\theta' \, dr' \, r' \, e^{i|\vec{k}-\vec{k}'|r'\cos\theta'} e^{-\kappa r'}$$

let $|\vec{k}-\vec{k}'| \equiv q$

$$= -\frac{mV_0}{\hbar^2} \int_0^\infty dr' \, \frac{r'}{iqr'} \, 2i\sin qr' e^{-\kappa r'}$$

$$= \frac{-2mV_0}{q\hbar^2} \int_0^\infty dr' \, \sin qr' e^{-\kappa r'}$$

$$= \frac{-2mV_0}{q\hbar^2} \cdot \frac{q}{q^2+\kappa^2}$$

$$= \frac{-2mV_0}{\hbar^2[\kappa^2+q^2]}$$; $q = k\sin^2\theta/2$, θ = scattering angle.

$$|f_k|^2 = \frac{4m^2V_0^2}{\hbar^4} \cdot \frac{1}{(\kappa^2+q^2)^2}$$

$$\sigma = \frac{4m^2V_0^2}{\hbar^4} \int d\cos\theta \, \frac{1}{(\kappa^2+q^2)^2}$$

(ii)

$d=1,\ V = V_0 e^{-\kappa r}$

$$\langle k'|T|k\rangle = \langle k'|V|k\rangle + \langle k'|V\frac{1}{E-H_0}V|k\rangle + \cdots$$

① $\langle k'|V|k\rangle = \int dx_1 \langle k'|x_1\rangle\langle x_1|V|k\rangle$

$$= \int dx_1\, e^{-ik'x_1}\, V(x_1) e^{ikx_1}$$

$$= V_0 \int dx_1\, e^{-[\kappa - i(k-k')]x_1}$$

$$= \frac{-V_0}{-i(k-k')+\kappa}$$

② $\langle k'|V\frac{1}{E-H_0}V|k\rangle$

$$= \int dx_1 dx_2\, e^{-ik'x_2} V_0 e^{-\kappa x_2}\left[-\frac{im}{\hbar^2 k} e^{-ik(x_1-x_2)}\right] V_0 e^{-\kappa x_1} e^{ikx_1}$$

$$= \int dx_1 dx_2\, e^{-x_2(ik'+\kappa-ik)}\, V_0^2\left(\frac{im}{-\hbar^2 k}\right) e^{-\kappa x_1}$$

$$= \left(-\frac{im}{\hbar^2 k}\right) V_0^2 \left(\frac{-1}{\kappa}\right)\left(\frac{-1}{\kappa - i(k-k')}\right)$$

$$\langle k'|T|k\rangle = \frac{-V_0}{\kappa - i(k-k')}\left[1 + \left(\frac{im}{\hbar^2 k^2}\right)V_0 + \cdots\right]$$

if $\left|\frac{im}{\hbar^2 k^2}V_0\right| \ll 1$,

$$= \frac{-V_0}{\kappa - i(k-k')} \cdot \frac{1}{1 - \frac{im}{\hbar^2 k^2}V_0}$$