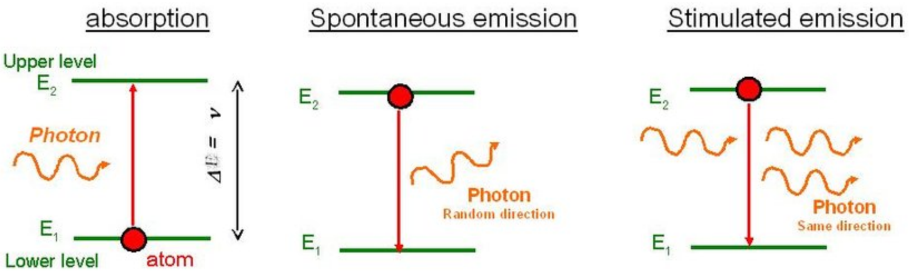
The PHOTON GAS

The absolutely crucial difference between (i) a gas of ^4He atoms or, eg., massive mesons and (ii) a gas of photons is that, while all of these are bosons, the photons are massless (they have no rest mass, and their dispersion relation is $\omega = ck$ (with energy $E = h\omega/2\pi$). This leads to a really important result – we know from thermodynamics that

$$\mu = \left(\frac{\partial U}{\partial N}\right)_{S,V} \quad \text{or that} \quad \mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} \quad \text{or that} \quad \mu = \left(\frac{\partial G}{\partial N}\right)_{T,p}$$

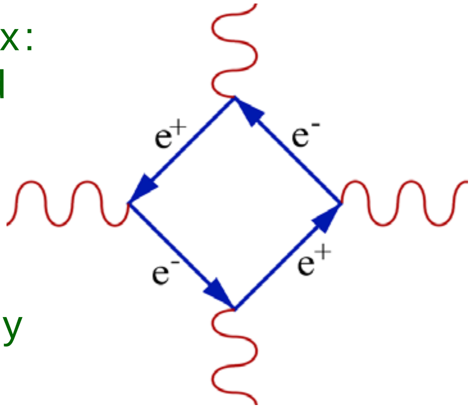
ie., no matter which variables we hold constant, μ measures the energy to add a particle



But a key fact about the photon gas is that in equilibrium with, eg., a set of atoms, the photon number is arbitrary; one can change the number of photons without changing the total energy.

Thus at equilibrium (where one of these thermodynamic potential is minimized), we have $\mu = 0$, and N is completely undetermined.

This is true even if for a set of photons inside a very large box: the box itself is made of matter, & so photons can be created or destroyed at the walls. To decouple photons from matter we need a perfect vacuum (ie., intergalactic space). The photon number is then conserved – it can only change via photon-photon interactions, which require creation of a very high-energy e^+e^- pair, with exponentially small probability (of order $\exp[-2mc^2/kT]$). Only then we can have non-zero μ



To discuss the thermodynamics of the photon gas we need to know (i) the density of states and (ii) the probability distribution function.

DENSITY of STATES: Let's put photons in a 3d box with sides L. We assume no electric charge in the box, so that the classical electric field obeys

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \quad \text{(empty vacuum)} \qquad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0 \quad \text{(wave equation)}$$

Now let's choose boundary conditions such that the components of E along the surface (tangential to the surface) go to zero at the surface S_B, ie., we have $\mathbf{k} \times \mathbf{E}_k(\mathbf{r}) = 0|_{\mathbf{r}\in S_B}$ on this surface, for the electric field component Ek with wave vector k. Since for an EM wave in vacuum we also have $\mathbf{k} \cdot \mathbf{E}_k(\mathbf{r}) = 0$ (because $\nabla \cdot \mathbf{E} = 0$), we then have the standing wave solutions

$$E_x = E_{x0} \sin \omega t \cos(k_x x) \sin(k_y y) \sin(k_z z)$$
$$E_y = E_{y0} \sin \omega t \sin(k_x x) \cos(k_y y) \sin(k_z z)$$
$$E_z = E_{z0} \sin \omega t \sin(k_x x) \sin(k_y y) \cos(k_z z)$$

with $\vec{k} = \left(\frac{m\pi}{L}, \frac{n\pi}{L}, \frac{l\pi}{L}\right)$

and $\left(\frac{\omega}{c}\right)^2 = \left(\frac{\pi}{L}\right)^2 (m^2 + l^2 + n^2) = k^2$

It is quite common in this business to also define the frequency in "cycles per second" (as opposed to Hz), and we will follow this, and write $\nu = ck/2\pi$ for the frequency in these units.

We then get the total number of states up to a frequency ν as

$$G(\nu) = 2\frac{1}{8}\frac{4\pi k^3/3}{(\pi/L)^3}\frac{1}{L^3} = \frac{1}{3\pi^2}\left(\frac{2\pi\nu}{c}\right)^3$$

so the density of states is $g(\nu) = \frac{dG}{d\nu} = \frac{8\pi\nu^2}{c^3}$

DISTRIBUTION FUNCTION FOR PHOTONS: We can assume that this will be the Bose distribution for a set of particles with $\mu = 0$, ie we have

$$n(E) = \frac{1}{e^{\beta E} - 1}$$ (Bose distribution for massless photons)

This is also what we get by assuming that each photon is like an oscillator; then, as we have seen

$$\Xi_\nu = \sum_{n=0}^{\infty} \exp[-n\beta h\nu] = \frac{1}{1 - \exp[-\beta h\nu]}$$ (for single oscillator with frequency ν)

so that, as we found previously, $\langle n_\nu \rangle = \frac{1}{\exp[\beta h\nu] - 1}$

for each oscillator - which reminds us of acoustic phonons (as it should). It then follows that we can write the expectation value of the energy for each oscillator as

$$\langle E_\nu \rangle = \frac{1}{\Xi_\nu} \sum_{n=0}^{\infty} nh\nu \exp[-n\beta h\nu] = -\frac{1}{\Xi_\nu} \frac{\partial \Xi_\nu}{\partial \beta} = \frac{h\nu}{\Xi_\nu} \Xi_\nu^2 \exp[-\beta h\nu]$$

and so we also get $\langle E_\nu \rangle = \frac{h\nu}{\exp[\beta h\nu] - 1}$ (energy per photon mode)

Now let's go to the total photon gas, ie., the complete partition function. We find $\ln \Xi$ in the usual way, by summing over the logs of the partition functions for individual photons, & weighting with the density of states. This then gives (multiplying by the system volume V):

$$\ln \Xi = V \frac{8\pi}{c^3} \int_0^\infty d\nu \, \nu^2 \ln(1 - e^{-\beta h\nu})$$

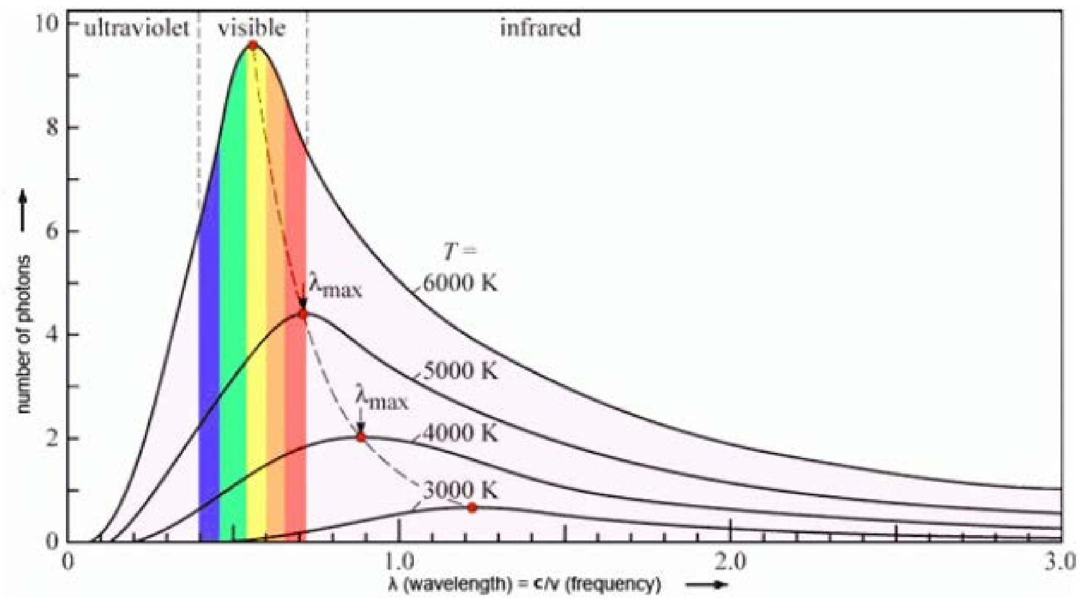
PLANCK DISTRIBUTION

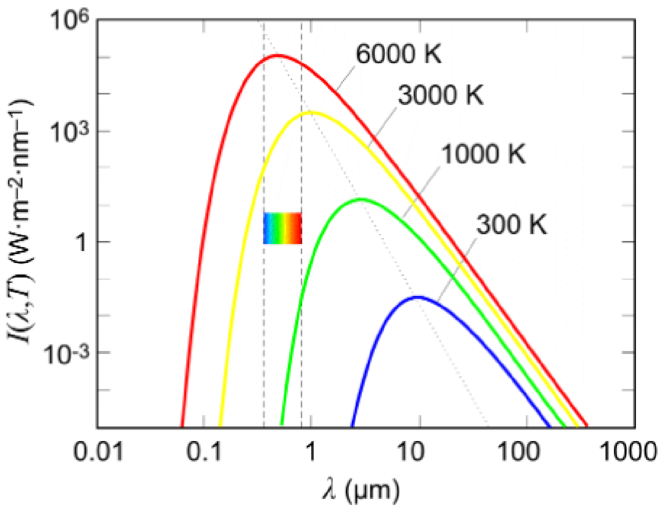
We can get the total energy by writing the energy per mode, and integrating over this, to get

$$\langle E \rangle = V \frac{8\pi}{c^3} \int_0^\infty d\nu \, h\nu^3 \frac{1}{\exp[\beta h\nu] - 1}$$

The energy density (E per unit volume) then has a contribution at frequency ν of

$$u_s(\nu, T) = h\nu \frac{8\pi\nu^2}{c^3} \frac{1}{\exp[\beta h\nu] - 1} = \frac{8\pi (k_B T)^3}{h^2 c^3} \frac{x^3}{\exp x - 1}$$ (PLANCK DISTRIBUTION)



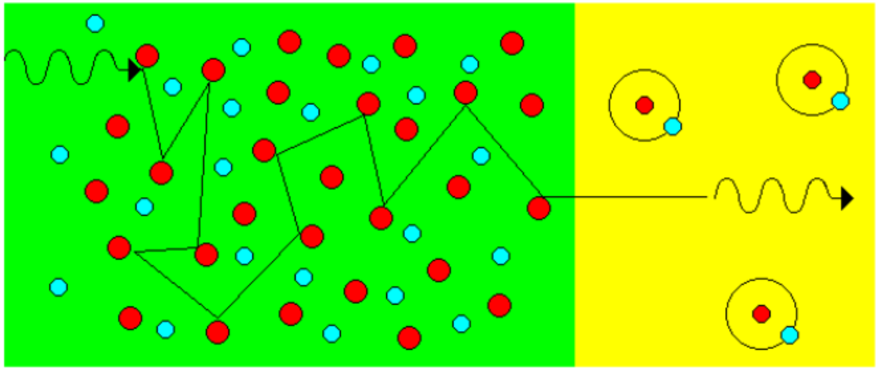


This famous distribution is shown here in 2 different ways

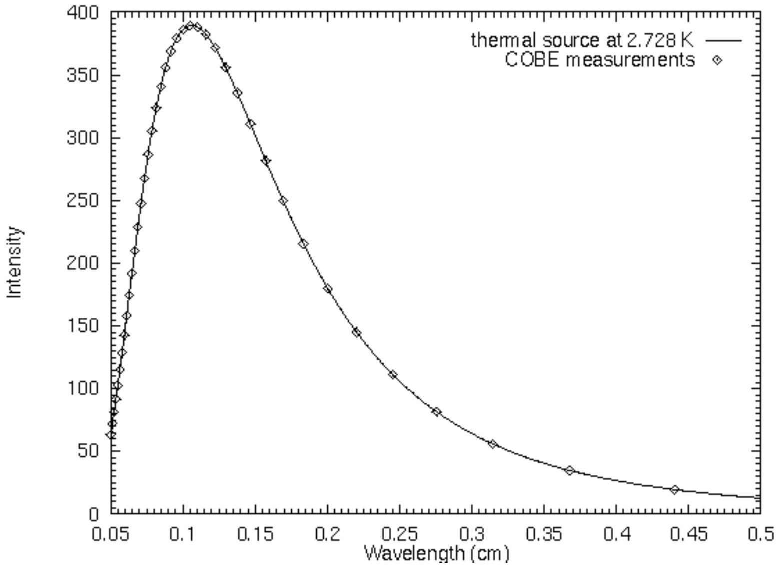
The total energy density of a photon gas at temperature T is then

$$u(T) = \int_0^\infty u(\nu, T) d\nu = \frac{8\pi (k_B T)^4}{h^3 c^3} \int_0^\infty \frac{x^3}{\exp x - 1} dx = \frac{8\pi (k_B T)^4}{h^3 c^3} \frac{\pi^4}{15} = \frac{8\pi^5 k_B^4}{15 h^3 c^3} T^4$$

PHOTONS in the UNIVERSE



In 1964 it was found that the universe was filled with photons at a temperature T ~ 2.728 K (from fits to the Planck curve); this is the "microwave background".

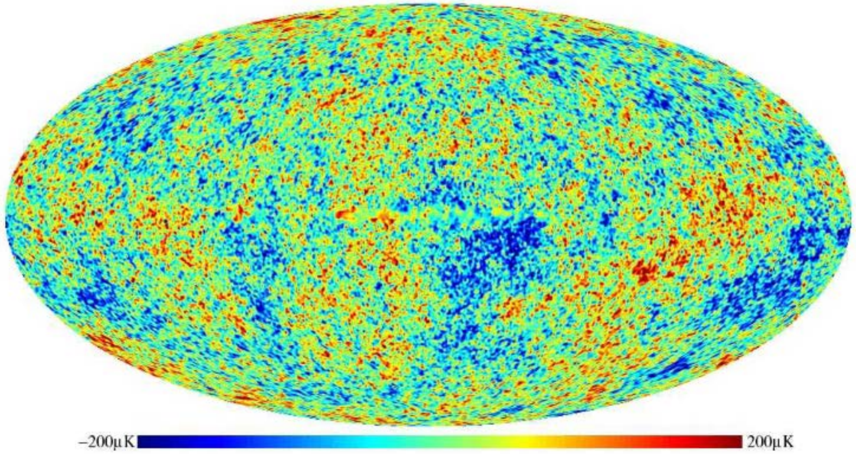


As the universe cooled and expanded after the Big Bang, a point was reached at which nucleons & electrons condensed out to form atoms - first for He and then H. The H condensation happened fairly suddenly, 360,000 yrs after the Big Bang.

The universe then became transparent to photons, which could no longer scatter off charges. The "stretching" of spacetime since then has red-shifted and cooled the photons.

The radiation is not exactly uniform – it shows relative fluctuations in intensity ~ 10^{-5}, which correspond to different photon and matter densities at the moment of their decoupling.

These fluctuations – measured with increasing accuracy in recent years – give detailed info about the Big Bang. some that it substantiates the "inflation theory" (universe appeared by Q tunneling)



MORE on the PLANCK DISTRIBUTION

LIMITING FORMS: The low and high-n limiting forms are interesting. Consider first the 19th century low-energy "Rayleigh-Jeans" form, viz.:

$$u_s(\nu,T) = \frac{8\pi h}{c^3}\frac{\nu^3}{\exp[\beta h\nu]-1} \xrightarrow{\text{Low frequency}} \frac{8\pi h}{c^3}\frac{\nu^3}{1+\beta h\nu-1} \sim \frac{8\pi\nu^2}{c^3}k_B T \qquad (h\nu \ll k_B T)$$

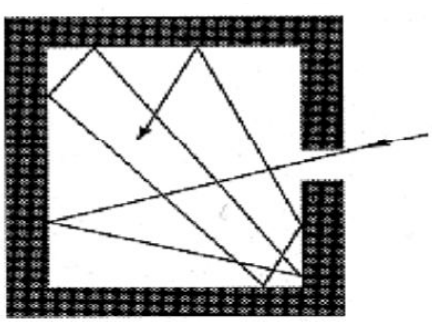
This form is classical - no factor of h appears in it, and it can be derived just by counting EM modes in a classical theory. However at high frequency it blows up, giving the famous 19th century "UV catastrophe".

The UV catastrophe was fixed by Planck, who introduced the idea of quantized energies to get the Planck distribution. From above, we see that the Planck form gives

$$u_s(\nu,T) \xrightarrow{\text{high frequency}} 8\pi h\frac{\nu^3}{c^3}e^{-h\nu/k_B T} \qquad (h\nu \gg k_B T)$$

This last form is the "Wien" form, also found long before QM was discovered.

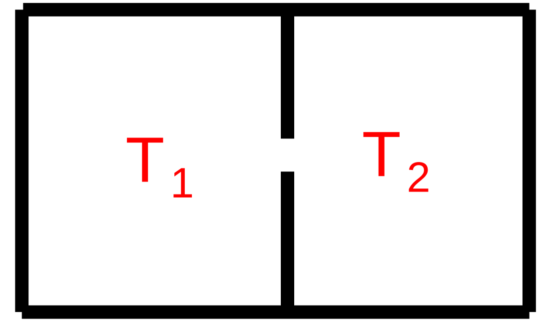
BLACK-BODY RADIATION: Radiation of the Planck form is often called "black body radiation". In fact, Planck, Kirchhoff, and Wien all came to their results by thermodynamic analysis, using "thought experiments".



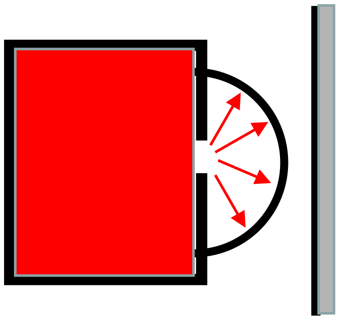
We imagine a system which is a perfect absorber – any radiation impinging on it never comes out again. An example which tends to this behaviour as the size of the hole goes to zero is the "black box" at left.

We define the emissivity $e_r(\nu, T)$ & absorptivity $\alpha_r(\nu,T)$ as functions of frequency and temperature. For a black box the absorptivity is unity for all frequencies and temperatures.

Consider 2 black boxes at 2 different temperatures T_1 and T_2. They can only be in mutual equilibrium if $T_1 = T_2 = T$. By using filters at different frequencies, this equilibrium can only hold if the energy absorbed at any frequency by one box equals that emitted by the other at the same frequency. Hence a perfect absorber is also a perfect emitter, for all frequencies, and indeed



$$e_r(\nu, T) = \alpha_r(\nu, T)$$ (for all frequencies)



Now let's consider the radiation emitted from the black box hole in the time interval dt, passing through a plane surface facing the hole. This first passes through a shell at distance $r = ct$ from the hole. We assume a hole of unit area and a shell of unit radius. Then we can calculate J_u, the total radiative energy/sec passing through the shell, ie., $J_u = dE/dt$, as follows:

We note that the energy density of the radiation in the box is just $u(T) = \dfrac{8\pi^5 k_B^4}{15h^3c^3}T^4$

This energy is leaving through the hole at a velocity $c\cos\theta$ towards the plane surface, so the total current is given by integrating this over the solid angle on the shell through which this radiation passes. This gives

$$J_u = \frac{cu(T)}{4\pi}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{2\pi}d\phi$$ (since the solid angle is just $d\Omega = \sin\theta\, d\theta\, d\phi$)

Hence we get $J_u = \dfrac{cu(T)}{4\pi}\left[\dfrac{\cos^2\theta}{2}\right]_{\pi/2}^{0} 2\pi = \dfrac{cu(T)}{4}$ ie. $J_u = \dfrac{2\pi^5 k_B^4}{15h^3c^2}T^4 = \sigma_B T^4$

(Stefan-Boltzmann law)