LECTURES 1-3 (Wed 12 Jan, Fri 14 Jan, Mon 17 Jan 2022)

PROBABILITY

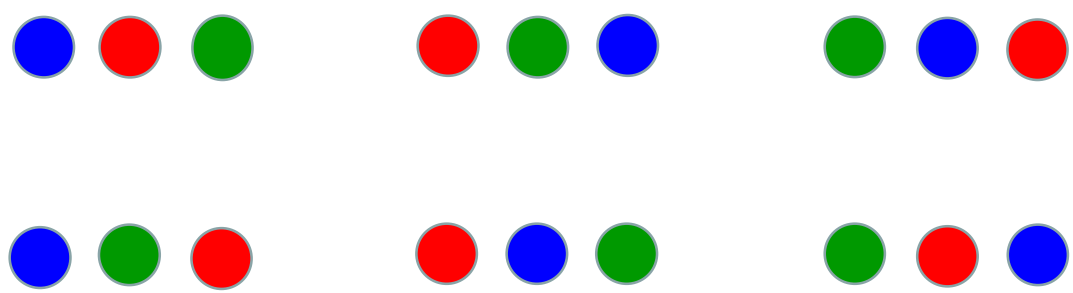
STEP 1: COUNTING STATES (COMBINATORICS)

N Distinguishable objects: **the definition is obvious. There are N! different possible arrangements (permutations)**

The object in the first position on the line may be chosen in N different ways, that in the second position in N−1 ways, and so on. The number of possible arrangements is therefore

$$N(N-1)(N-2)(1) = N!$$

Example: 3 distinguishable objects



At left we see the 6 different ways of ordering (ie., permuting) 3 different balls.

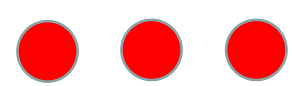
$\boxed{3! = 6}$

N Indistinguishable objects: **One makes one of 2 assumptions. Either:**

(i) the definition is absolute – there is no difference in principle, AT ALL, between the different permutations (as in quantum mechanics); or

(ii) We simply decide we don't want to distinguish them (as for coin tosses)

Then there is only ONE permutation

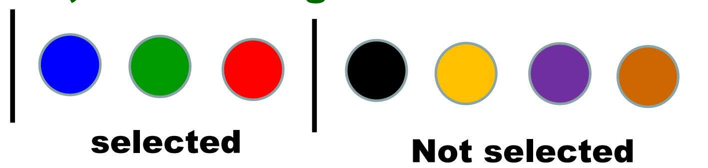


$\boxed{1! = 1}$

BINOMIAL DISTRIBUTION

Let's generalize the previous example with the 2 following examples

Example 1: We want to know the total number of ways of extracting n objects from N distinguishable objects, without regard for the order in which they are selected.



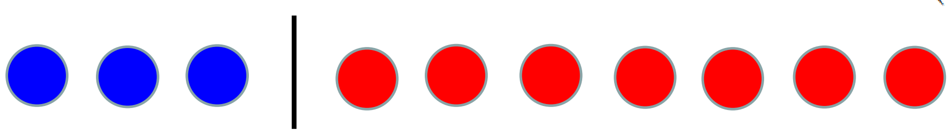
We can organize the N objects in N! different ways. However re-orderings of the n! objects in the group selected, and the (N-n)! objects in the remaining group not selected, do not count, since they do not change this. The number of possible arrangements is therefore

$$C_n^N = \frac{N!}{n!(N-n)!}$$ **(binomial factor)**

Example 2: We have 2 sets of indistinguishable objects; the total number is N, and n of them in one set, N-n in the other. What is the total number of distinguishable ways of ordering them? These could be, eg., H or T for N coins.

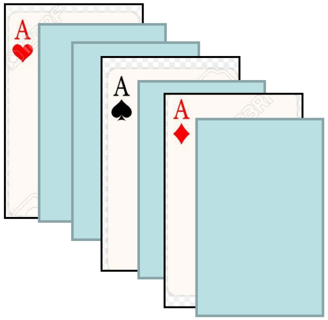
We can organize the N objects in N! different ways. However re-orderings of the n! objects in the 1^{st} set, and the (N-n)! objects in the 2^{nd} set, do not count, since all these different rearrangements are indistinguishable from each other. The number of possible arrangements is therefore again

$$C_n^N = \frac{N!}{n!(N-n)!}$$ **(binomial factor)**



Suppose, eg., I toss 10 coins, and I want to know how many different arrangements of these 10 tosses will have 3 heads and 7 tails turn up, in any order. The answer is 10!/(3! 7!) = 120

BINOMIAL DISTRIBUTION (Cont.)



Example 3: A card game.

Suppose we are dealt 7 cards from a 52-card pack. What is the probability this hand contains 3 Aces?

To do this we need to first ask how many possible outcomes there are for the 7 cards that are dealt; we then ask how many of these give 3 Aces.

(1) Total number of possible distinguishable arrangements is the binomial $C^{52}_7 \equiv C^{52}_{45} = 52!/(7!\ 45!)$ This is because we can re-order the first 7 cards 7! times, and the last 45 cards 45! times

(2) To find how many of these are Aces, we note first that it does not matter which Aces we get. We need to multiply the number of ways of getting 3 of the 4 Aces (without caring which ones), by the total number of outcomes for the other 4 cards that are dealt, with the constraint that these other cards are NOT Aces. The first number is $C^4_3 = 4$.To find the second number, we note that there are 48 cards that are not Aces, and we are getting 4 of these. So this latter number is $C^{48}_4 = 48!/(44!\ 4!)$

The final result for the probability $P^{\{7\}}_{AAA}$ is then

$$P^{\{7\}}_{AAA} = \frac{C^4_3 C^{48}_4}{C^{52}_7} = 4 \times \frac{48!}{4!\ 44!} \times \frac{7!\ 45!}{52!} = 7.6.5.4 \frac{45}{52.51.50.49}$$

which if we work it out gives $P^{\{7\}}_{AAA} \sim 0{:}00582$, ie., roughly 1/172.

STEP 2: ASSIGNING PROBABILITIES

We've found the number of permutations – now we must assign a probability to each one.

The simplest is to assign equal probabilities to each permutation or "outcome". If the total number of permutations is X, the probability of each outcome is 1/X

Example 1: Coin tossing

We look for the probability of getting n heads (ie., spin up) from N tosses. Suppose the coins are equally balanced. The total number of possible outcomes is 2^N; each therefore has probability $1/2^N$. However the total number of permutations having n heads is C^N_n.

We then have $$P_N(n) = \frac{C_n^N}{2^N} = \frac{1}{2^N}\left(\frac{N!}{n!(N-n)!}\right)$$

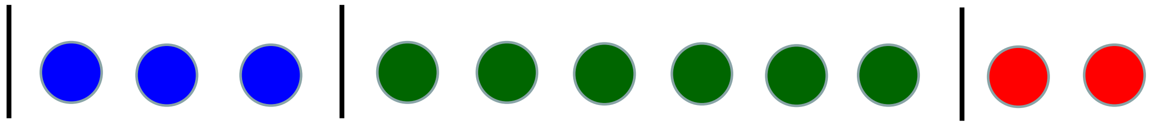
Unequal Probabilities: **Suppose coins are unbalanced; the probability of getting a head (spin up) is p_+, and that of getting a down is $p_- = 1- p_+$. Then we have**

$$P_N(n) = C_n^N \, p_+^n p_-^{N-n)} \equiv \left(\frac{N!}{n!(N-n)!}\right) p_+^n (1-p_+)^{(N-n)}$$

because the probability of getting any one of the combinations with n heads and N−n tails is, by assumption, just the product over the probabilities for each throw. When $p_+= 1/2$, this just reduces to the previous result.

MULTINOMIAL COMBINATORICS

Here we partition the N objects into m different groups. We want to know the number of permutations/outcomes in which internal permutations of objects inside a box are considered to be indistinguishable



n_1 objects (n_1 = 3) **n_2 objects (n_2 = 6)** **n_3 objects (n_3 = 2)**

By the same arguments as before we have:

$$C^N_{\{n_j\}} \equiv C^N_{n_1,n_2,..n_m} = \frac{N!}{n_1!n_2!....n_m!}$$ **permutations**

since the k-th group of identical objects can be rearranged in $n_k!$ ways without changing anything, and we can do this for any of the m different sub-groups.

Example 1: We throw N dice, each having six possible states (1-6) What is the total number of outcomes with n_1 showing 1, n_2 showing 2, etc, with $\Sigma_i n_i = N$?

The answer is $C^N_{n_1,n_2,..n_m}$ **with m=6.**

Example 2: a set of N particles, each with spin s. For each spin, there are m = 2s + 1 different distinguishable states (spin projections) for each spin. What is the total number of different possible combinations with n_1 spins having spin projection s_z = s, n_2 with projection s-1, etc, up to n_m with spin projection s_z = -s.

The answer is again $C^N_{n_1,n_2,..n_m}$ **with m=2s + 1.**

MULTINOMIAL PROBABILITIES

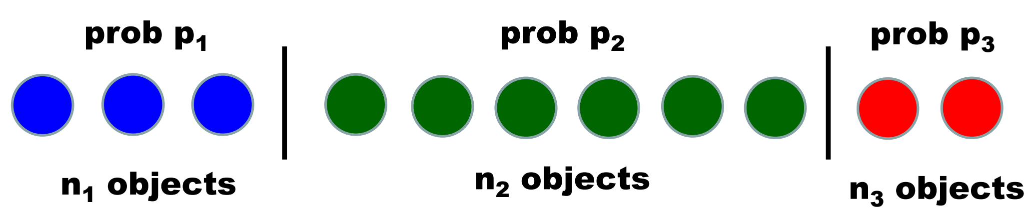
We've now done the combinatorics for the multinomial distribution – we must again assign a probability to each one.

Again, the simplest is to assign equal probabilities to each permutation/outcome. If the total number of permutations is X, the probability of each is again 1/X

Suppose we have N balls which we distribute in m different cells or boxes, but now the probability of going into the k-th box is p_k, where k= 1;2;···m (and where of course $\sum_k p_k = 1$). As we saw before, the number of different ways of doing this is just the multinomial coefficient

$$C^N_{n_1, n_2, .. n_m} = \frac{N!}{n_1! n_2! .... n_m!}$$

However now the weighting attached to any one of these ways is: $\Pi_k (p_k)^{n_k}$



It then follows that the probability $P_N(n_1; n_2; \cdots n_m)$ of getting an outcome in which there are n_k balls in the k-th box is just

$$\begin{aligned} P_N(n_1, n_2, \cdots n_m) &= C^N_{n_1, n_2, .. n_m} \prod_{k=1}^{m} p_k^{n_k} \\ &= \delta(N - \sum_k n_k) \left(\frac{N!}{n_1! n_2! n_m!} \right) p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m} \end{aligned}$$

Kronecker Delta Function

MULTINOMIAL DISTRIBUTION (Cont)

Example 3: **Another card game.**

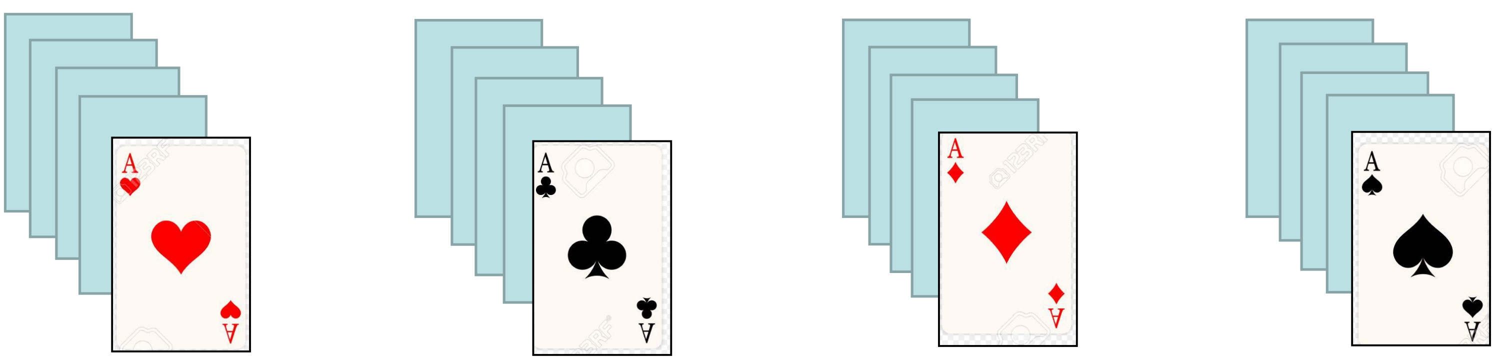
Suppose we have 4 players, & each one is dealt 5 cards. What is the probability that each player has exactly one Ace?

This generalizes the previous card problem to a multinomial distribution. We must first ask how many possible outcomes there are for the 4 batches of 5 cards that are dealt; we then ask how many of these give 1 Ace in each hand.

(1) There are $C^{52}_{5:5:5:5:32} \equiv 52!/[(5!)^4 32!]$ ways of distributing the cards amongst 4 hands of 5 cards, and then amongst the remaining 32 cards.

(2) There are 4! ways of ordering the 4 Aces. There are then 48 cards left, that are not Aces - these can be dealt out to the 4 different hands in a total of $C^{48}_{4:4:4:4:32} \equiv 48!/[(4!)^4 32!]$ times.

Probability is then $$P^{\{4\times5\}}_{4A} = \frac{4! \times C^{48}_{4.4.4.4.32}}{C^{52}_{5.5.5.5.32}} = \frac{5^4 \times 24}{52.51.50.49} \sim 2:31\times10^{-3} \sim 1/433$$



See Notes and Homework assignments for other examples