INDISTINGUISHABILITY in QUANTUM MECHANICS

There is a key fact about Quantum Mechanics that one just has to accept. It is that if a number of different alternatives are allowed, then QM permits one to sum over these in finding the QM STATE or "AMPLITUDE". This is written as

$$|\Psi> \;=\; \Sigma\, c_k\, |\phi_k> \;=\; \Sigma\; a_k \exp\,[i\theta_k\;]\, |\phi_k>$$

where the coefficients for a simple particle are just complex numbers, and where the probability associated with this state is the "Amplitude Squared"

$$P_\Psi \;=\; <\Psi\,|\,\Psi> \;=\; |\Psi|^2 \;=\; \Sigma\,|c_k\,|^2 \;=\; \Sigma\, a_k^{\;2}$$

so that for a normalized state, in which we know that the system is in state $|\Psi>$, we have

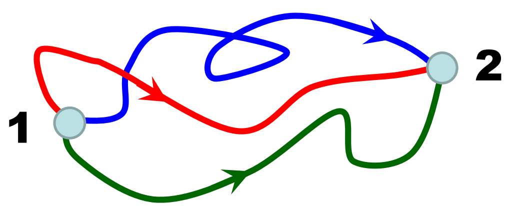
$$P_\Psi \;=\; |\Psi|^2 \;=\; \Sigma\,|c_k\,|^2 \;=\; 1$$

This is the famous "superposition principle".

In a development you may not have heard of, Feynman pointed out that, during the time evolution of a quantum system between 2 states $|\Psi_1>$ and $|\Psi_2>$, one had to SUM OVER ALL ALLOWED PATHS between these states, to find the transition amplitude $K_o(2,1)$, where

$$K_o(2,1) \;=\; <\Psi_2(t_2)|\,\exp\,[-iH_o(t_2 - t_1)]\,|\Psi_1(t_1)> \;\rightarrow\; \int_1^2 \mathcal{D}\mathbf{r}(\tau)\; e^{\frac{i}{\hbar}S_{21}[\mathbf{r}(\tau)]}$$

Feynman's way of writing this



HOWEVER: as noted by Pauli, 2 or more identical particles in QM are INDISTINGUISHABLE in principle....

COMPARING INDISTINGUISHABILITY & DISTINGUISHABILITY

(1) DISTINGUISHABLE PARTICLES

Consider two ***distinguishable*** particles (with coordinates labeled by $\mathbf{r}_1 = 1$ and $\mathbf{r}_2 = 2$) occupying two single particle states φ_A and φ_B. Now if the particles are not in any way correlated, we can write down a state of form $\psi_{AB}(1,2) = \varphi_A(1)\,\varphi_B(2)$, in which the "first particle" (ie., the one at position $\mathbf{r}_1$) is in state φ_A, and the second one is in state φ_B.

NB: Here the colours indicate the state of the particle

(2) INDISTINGUISHABLE PARTICLES

Now suppose the particles are ***indistinguishable***. Then this means that, from the fundamental physical standpoint, there is no difference whatsoever between the state $\psi_{AB}(1,2) = \varphi_A(1)\,\varphi_B(2)$ and the state $\psi_{BA}(1,2) = \varphi_A(2)\,\varphi_B(1)$, in which we have swapped the particle coordinates between the states A and B.

is indistinguishable from the 'swapped' or 'exchanged' state

Then QM says that we must sum over both alternatives !!

QUANTUM STATES for 2 PARTICLES

What are the allowed states in QM, for a pair of particles? I will give 3 arguments to indicate that we can have fermions and bosons

Argument #1

This is an "old style" argument – it goes back to Pauli, and is the one found in most textbooks. It goes as follows:

We require the probability density $|\Psi|^2$ to be well defined. However, if we write a wave function like $\psi_{AB}(1,2)$ for the pair of states, this is not the case – for this state is supposed to be indistinguishable from $\psi_{BA}(1,2)$, which is supposed to represent the SAME physical state. Clearly however it is not, because $|\Psi|$ is ***not*** the same for the 2 states (to see this, suppose that state A is strongly localized around its coordinate, whereas state B is spread out around its coordinate – then clearly $|\psi_{AB}(1,2)|$ is not the same as $|\psi_{BA}(1,2)|$, as a function of the coordinates $\mathbf{r}_1 = 1$ and $\mathbf{r}_2 = 2$)

However there are 2 states that do fit the bill. To see this, note first that to satisfy the "exchange symmetry" relation $|\Psi(1,2)|^2 = |\Psi(2,1)|^2$, we can have either $\Psi(1,2) = \Psi(2,1)$, or $\Psi(1,2) = -\Psi(2,1)$. We then see that we can have one of two possible states for the pair wave-function involving one particle states φ_A and φ_B. These are

$$\Psi(1,2) = \frac{1}{\sqrt{2}}[\phi_A(1)\phi_B(2) + \phi_A(2)\phi_B(1)] \qquad (bosons)$$

Sum one state to the other

$$\Psi(1,2) = \frac{1}{\sqrt{2}}[\phi_A(1)\phi_B(2) - \phi_A(2)\phi_B(1)] \qquad (fermions)$$

Subtract one from the other

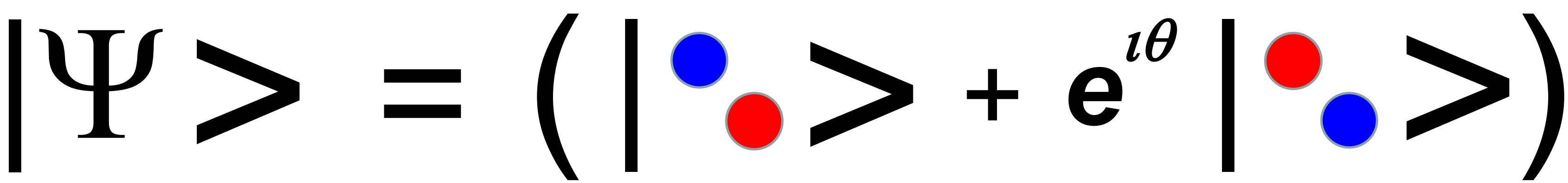
Argument #2

This is a faster way of getting the same result, and follows from the general idea that one should "sum over all allowed alternatives" It goes as follows.

We note that in QM, the total wave-function can be composed of superpositions, with arbitrary coefficients, of all possible states that are solutions of the relevant Hamiltonian with appropriate boundary conditions. In the present case, we only have 2 states A and B, and to get the right coefficients here we invoke the exchange symmetry requirement above – clearly a wave function of form

$$\Psi(1,2) \sim [\ \varphi_A(1)\ \varphi_B(2) + e^{i\theta}\ \varphi_A(2)\ \varphi_B(1)\ ]$$

will fit the bill, and then to satisfy the condition $|\Psi(1,2)|^2 = |\Psi(2,1)|^2$, we see that we must have either $\theta = 0$ or $\theta = \pi$. This then gives the previous result for fermions and bosons.



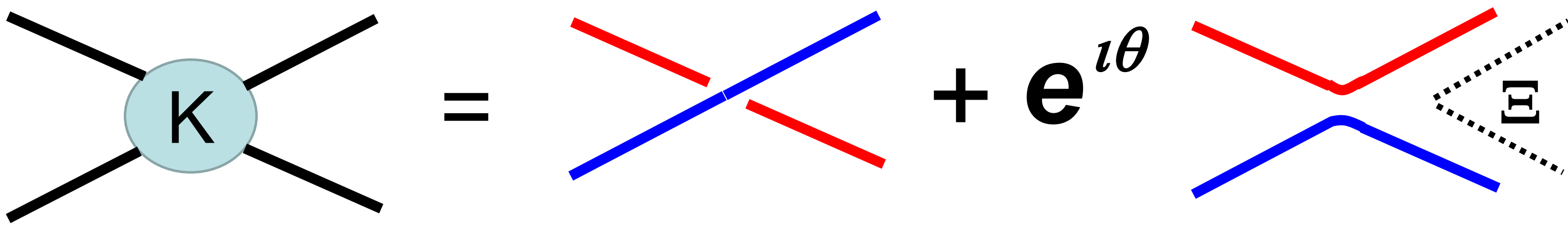
where $\theta = 0$ (bosons)

$\theta = \pi$ (fermions)

Argument #3

This one uses path integrals – I am only showing it for those who are interested. It goes as follows.

Imagine we scatter 2 identical particles, as shown in the figure.



In QM, we must sum over the 2 possible alternatives shown for the transition amplitude. Here we only have 2 choices: an amplitude K(0) where the scattering angle is 0, and an amplitude K(Ξ) for a scattering angle Ξ. Summing over these we get

$$K \sim [\ K(0) + e^{i\theta} K(\Xi)\ ]$$

and then the easiest way to get the previous result for fermions and bosons is to choose $\Xi = \pi$. Then, using the result $|K(0)|^2 = |K(\Xi)|^2$ for the transition probability, we see that we must have either $\theta = 0$ or $\theta = \pi$, ie., the previous result for bosonns and fermions. NB: this argument assumes 3 dimensions. In 2 dimensions one find that the `statistical angle' θ is arbitrary.

The key question for us in discussing <u>quantum statistical mechanics</u> is how one generalizes this 2-particle discussion to N particles. I will discuss this later in the course……