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PHYS 403: HOMEWORK ASSIGNMENT No. 3: QUANTUM GASES and SUPERFLUIDS

(Mar. 10th, 2022)

HOMEWORK DUE: WEDNESDAY, March 23rd, 2022

To be uploaded by 11.59 pm, Wednesday March 23rd - Late Homework will not be accepted

QUESTION (1) EARLY UNIVERSE: At the 'recombination time' τ_R (roughly 400,000 yrs after the Big Bang), the main constituents of the universe were photons, H atoms, protons, and electrons. Let's ignore the photons here, and assume that the 3 remaining species have chemical potentials μ_H, μ_p, and μ_e, and number densities n_H, n_p, and n_e, respectively. Assume a hydrogen ionization energy E_o, and that that there are 2 relevant states for the proton and electron (they are spin-1/2), and hence 4 states for the H atom.

1(a) Suppose we can treat this system as low density. Then what are n_H, n_p, and n_e in terms of μ_H, μ_p, and μ_e?

1(b) What defines thermal equilibrium for this system, and at equilibrium, what are n_H, n_p, and n_e?

1(c) Using values for E_o and for the mass m_e of an electron that you can get from the literature, find the density n_e when $n_H = n_p = n_e$ (ie., half the H atoms are ionized), which gives the density at the time τ_R.

QUESTION (2) BOSE GASES:

2(a): Draw two graphs as a function of energy E which shows (i) the 1-particle density of states, and (ii) the Bose distribution function, for a 3-dimensional Bose system of massive particles, for the cases $T > T_c$ and $T < T_c$. Here T_c is the BEC condensation temperature. Then draw two graphs showing the product of these 2 functions as a function of energy, again for these 2 cases.

2(b) A criterion for BEC to occur in a 3-d gas of bosons is that the chemical potential $\mu = 0$. Explain this criterion with reference to the relevant mathematical expressions.

2(c) Rederive the criterion for 2-d and 1-d systems. What do the results tell you about BEC in these cases?

2(d) Consider now the photon gas. Why is $\mu = 0$ always for photons? Now, derive an expression for the energy density $u(T)$ for a photon gas in n dimensions, where n is a positive integer; and show that $u(T) \propto T^{n+1}$.

QUESTION (3) SUPERFLUIDS

3(a) Suppose a mass M is moving through a fluid with constant viscosity coefficient η. Find the equation of motion of the particle, assuming there is an external force $f(t)$ acting on it. If the initial velocity at $t = 0$ is $v(t = 0) = v_o$, then show the solution to this equation of motion is

$$v(t) \;=\; v_o e^{-\gamma t} \;+\; \int_0^t dt' \, \frac{f(t')}{m} \, e^{-\gamma(t-t')}$$

where $\gamma = \eta/M$. Then show that if the force $f(t) = f_o$, a constant, then after a long time the particle will reach a constant velocity v_f; and find v_f.

3(b) In a superfluid the friction depends on the velocity. Suppose that $\eta(v) = \eta_o(v - v_c)\, \theta(v - v_c)$, where η_o is a constant, and $\theta(x) = 0$ for $x < 0$, and $\theta(x) = 1$ for $x > 0$. Find the new terminal velocity v_f, without solving the new equation of motion.

3(c) Superfluids have quantized vortex ring excitations. For a circular ring of radius R, the energy $E \sim \frac{1}{2}\rho\kappa^2 R \ln[R/a_o]$, and the momentum $p \sim \pi\rho\kappa R^2$, where ρ is the superfluid density, κ the circulation quantum, and $a_o \sim 0.1\ nm$ is a vortex core radius. If the critical velocity for formation of a vortex ring is $v_c \sim min(E/p)$, then show that in an infinite system, $v_c \to 0$; and also find v_c if the superfluid is moving through a cylindrical tube of radius R_o. Finally; since the vortex ring velocity is $v = dE/dp$, find $v(R)$ as a function of R, and sketch a graph of it.

END of 3RD HOMEWORK ASSIGNMENT