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PHYS 403: REVISION QUESTIONS (short) 2022

NOTES: The exam is divided into 6 short questions (in section A) and 4 long questions (in section B). You must answer a total of **THREE** short questions from section A, and **TWO** long questions from section B. You can choose which questions you decide to answer. Note that extra marks will not be given for answering more than 4 questions in section A or 2 questions in section B; if you do, we will simply choose those questions which give you the highest mark.

Here is a set of short questions. You should take **ROUGHLY** 20 mins to do any one of them (some of them may take a little longer). I will send out answers to them all in a few days.

SECTION A: SHORT QUESTIONS

QUESTION A.1: QUANTUM GASES

(i): Why does the diameter of a white dwarf decrease when its mass increases?

In a non-relativistic analysis, this is because the negative gravitational energy increases faster with mass (going as M^2) than the positive degeneracy energy increases (this goes as $M^{5/3}$, although you don't have to derive this). Hence increasing the mass decreases the radius at which the energy is minimized. Roughly speaking, the gravitational attraction becomes ever more important as M increases. The same is even more true in a relativistic analysis.

(ii) Why does the chemical potential of a gas (Bose, Fermi, or classical) never increase (and almost always decreases) as one raises the temperature?

Roughly speaking this is because the free energy $F = U - TS$ is governed by the behaviour of the entropy when T is large (and also when the density of the system is low); in both these cases the occupation numbers of the states are low. The entropy is increased by allowing the system to spread over more states. Since the chemical potential $\mu = (\partial F / \partial N)$, and adding a particle will cause an increase in entropy, we get $\mu < 0$.

One can see this purely mathematically as well, by examining the usual equation relating N to μ for a Fermi or Bose gas; at high T this can only be satisfied by allowing μ to decrease relative to the $T = 0$ value.

There are other ways to explain this as well.

QUESTION A.2: 2-LEVEL SYSTEMS

(i): Consider a set of N non-interacting 2-level systems (TLS), with level energies E_1 and E_2 for each of the TLS. At temperature T , what is the average energy $U(T)$ for the total system? Derive also the specific heat $C_V(T)$.

The average energy for the system is easy, since there are only 2 levels; we have

$$U(T) = N \frac{E_1 e^{-\beta E_1} + E_2 e^{-\beta E_2}}{e^{-\beta E_1} + e^{-\beta E_2}} = N \frac{E_1 + E_2 e^{-\beta \Delta_o}}{1 + e^{-\beta \Delta_o}}$$

where $\Delta_o = |E_1 - E_2|$. The specific heat is easily found from this to be

$$C_V(T) = N \left(\frac{\Delta_o}{kT} \right)^2 \frac{e^{-\Delta_o/kT}}{(1 + e^{-\Delta_o/kT})^2}$$

(ii) Find expressions for $U(T)$ and $C_V(T)$ when $kT \gg |E_1 - E_2|$. You should find the $T = \infty$ result, and also the first correction to this result, for finite (but very large) T .

The high- T limiting result for the energy is obtained by expanding the exponent for small β ; the result is

$$U(T) = N \left[\frac{1}{2}(E_1 + E_2) - \frac{\Delta_o^2}{4kT} \right] \quad (T \rightarrow \infty)$$

and this just gives the specific heat in this limit as

$$C_V(T) = \frac{N}{4} \left(\frac{\Delta_o}{kT} \right)^2 \quad (T \rightarrow \infty)$$

QUESTION A.3: FERMI DISTRIBUTION

(i): The grand canonical partition function for a single fermion state of energy ϵ is $z(\epsilon) = \sum_n \exp[n\beta(\mu - \epsilon)] = 1 + \exp[\beta(\mu - \epsilon)]$. Show that the mean occupation number for this state is just the Fermi function, ie., that $\langle n \rangle \rightarrow f(\epsilon - \mu) \equiv \{1 + \exp[\beta(\epsilon - \mu)]\}^{-1}$, which we also write as $f(x) = [1 + e^{\beta x}]^{-1}$, where $x = (\epsilon - \mu)$.

This is a standard exercise, done in the course notes. We write for the probability $P(n) = z^{-1} e^{\beta n(\mu - \epsilon)} = z^{-1} e^{n\beta x}$, and then the expectation value $\langle n \rangle = z^{-1} \sum_n n P(n)$, which since $n = 0$ or $n = 1$ is just

$$\langle n \rangle = \frac{1}{[1 + \exp[\beta(\epsilon - \mu)]]}$$

(ii) Then show that the probability of finding n particles in this state is

$$P(n) = \frac{[1 - f(-x)]^n}{[f(-x)]^{n-1}}$$

This is a little finicky because of the “-” signs. We have $P(n) = z^{-1} e^{\beta n(\mu - \epsilon)} \equiv e^{-n\beta x} / [1 + e^{-\beta x}]$. Now compare with the expression given above

$$P(n) = \frac{(1 - f(-x))^n}{f(-x)^{n-1}} \rightarrow \frac{e^{-n\beta x}}{1 + e^{-\beta x}} \quad QED.$$

QUESTION A.4: INTERATOMIC POTENTIAL

(i): Consider the 1-dimensional potential

$$V(x) = V_o \left[\left(\frac{a_o}{x} \right)^{12} - 2 \left(\frac{a_o}{x} \right)^6 \right]$$

Find the value of x for which $V(x)$ is a minimum, and find the “curvature” d^2V/dx^2 at this point. What is the frequency of small oscillations of a particle of mass M about the minimum in this potential?

The minimum of the potential, given by the value of x for which $V(x)$ is minimized, is at $x = a_o$. The 2nd derivative is

$$\frac{d^2V}{dx^2} = \frac{12V_o}{x^2} \left[13 \left(\frac{a_o}{x} \right)^{12} - 7 \left(\frac{a_o}{x} \right)^6 \right] \Big|_{x=a_o} \rightarrow 72 \frac{V_o}{a_o^2}$$

This SHO potential then has a small oscillation frequency ω_o , for a mass M , given by $\omega_o^2 = 72(V_o/a_o^2 M)$.

(ii) Draw a picture of the potential $V(x)$, and explain briefly how it can be used to model interatomic interactions. For such interaction, what do you think are typical values for V_o and a_o ?

The picture is straightforward. For the size of the constants, the key here is to point out that the short range repulsion is from the Pauli principle, and the long-range attraction is van der Waals; such forces exist between all atomic clouds. The numbers here are just part of general knowledge about matter in the physical world. All atoms have more or less the same size, of order Angstroms; thus a_o is one or two Angstroms. The size of V_o is more difficult. All chemical energies are of order a few eV. However we notice that V_o is just the depth of the potential well, where the Pauli and van der Waals energies balance. We can guess that this energy is roughly that involved in making a simple system stay liquid or solid, as opposed to evaporating. This would then give $V_o \sim 100 K$, ie., $V_o \sim 10^{-3}$ eV.

QUESTION A.5: ARGON in ATMOSPHERE

(i): Roughly 1 percent of the volume of the earth's atmosphere is composed of ^{40}Ar . Suppose you are in a bedroom with a volume of 60 m^3 . Roughly how many ^{40}Ar atoms are in the room, and what is their total mass?

NB: This question was extracted from an exam question from last year, but it was a TAKE HOME exam - so studentds had access to the internet, the values of constants, etc. I would never set a question in this form in an in-person exam, where you waon't have these numbers available.

I did say that you should estimate this "roughly". So you could say something like the following:

" 10^3 kg of water, with molecular weight 18, occupies 1 m^3 . Atmospheric pressure supports a column of water roughly 10m high (this is easy to show - atmospheric pressure is $\sim 10^5 \text{ Nm}^{-2}$ and $g \sim 10 \text{ ms}^{-2}$, so atmospheric pressure is equivalent to a weight $\sim 10^4 \text{ kgm}^{-2}$). The scale height of the atmosphere is $\sim 5 \text{ km}$, so the atmospheric pressure is equivalent to that coming from a 10 km column of air at atmospheric density. Thus 1 kg of air occupies roughly 1 m^3 . The atomic weights of N_2 and O_2 molecules are 28 and 32 respectively, so the mass fraction of air coming from ^{40}Ar is roughly $4/3 \times 0.01 \sim 0.013$, with total mass $\sim 13\text{g}$, ie., roughly $1/3$ mole of ^{40}Ar . Since Avogadro's number is 6×10^{23} , this implies we have roughly 2×10^{23} atoms of ^{40}Ar per cubic metre. Multiplying by 60, we then get a number $N \sim 1.2 \times 10^{25}$ atoms in the bedroom, with mass of roughly 0.78 kg ."

There will be many ways to get this rough answer, so I won't try to guess them all.

(ii) In MKS units, roughly what is the total thermal energy associated with the ^{40}Ar atomic motion?

The energy per atom is $3k_B T/2$. Let's assume room temperature (ie., $T \sim 300\text{K}$). Then, using $k_B = 1.38 \times 10^{-23} \text{ m}^2 \text{ kg s}^{-2} \text{ K}^{-1}$, we have a total thermal energy in the room given by $E = 3Nk_B T/2 \sim 7.45 \times 10^4 \text{ J}$, at room temperature.

QUESTION A.6 BLACK BOXES:

(i) Suppose I surround the sun (considered to be a black body at a temperature T_s) with a perfectly black shield. Assuming the shield is indestructible, it will come to an equilibrium temperature T_B , governed by the radiative energy flow from the sun and the energy radiated by the shield. Assuming that the universe outside the shield can be treated as a reservoir at temperature $T = 0$, show that the equilibrium temperature of the shield is $T_B = (T_s/2)^{1/4}$. Assume the shield is close to the sun's surface, so that the surface area of the shield is the same as that of the sun.

(ii) Now suppose instead I replace the single shield with 2 concentric shields. What is the temperature T_1 of the first (inner) shield, and T_2 for the second outer shield?

(ii) Finally let us generalize the argument to a set of N concentric black shields. What is the temperature T_n of the n -th shield, for $1 \geq n \geq N$?

In this problem, suppose the radiative energy flow (a power) from the sun, per unit area, is P_s , and the energy flow from the shield, per unit area (inwards and outwards) is P_B in each direction. Then the net flow from sun to shield is $P_s - P_B$, and the net flow out from the shield is P_B . For equilibrium these have to be the same, so we have $P_s = 2P_B$. Then from the Stefan-Boltzmann law, according to which $P \propto T^4$, it follows that $T_B = T_s 2^{-1/4}$. This answers (i).

For (ii), we simply extend the above argument. We define powers P_j , with $j = 1, 2$ emitted in each direction by the shields, and a power P_s as before from the sun. We then go through the same argument, to find that $P_s - P_1 = P_1 - P_2 = P_2$. This gives us $P_s = 3P_2$, and $P_1 = 2P_2$, and so we get $T_2 = T_s 3^{-1/4}$ and $T_1 = T_s (2/3)^{1/4}$.

The problem in (iii) is just a simple inductive generalization of the above.

QUESTION A.7 FREE ENERGY: For a gas of particles, the infinitesimal changes dS in the entropy and dV in the volume of the gas container result in a change dU in the energy, given by $dU = TdS - pdV$, where T is the temperature and p the pressure in the gas.

(i) The free energy of the system is $F = U - TS$. Find an expression for the infinitesimal dF , and show that the pressure is then given by the partial derivative $p = -(\partial F / \partial V)|_T$, where T is held constant.

(ii) Suppose we are allowed to add particles to the gas as well, so that $dU = TdS - pdV + \mu dN$, where μ is the chemical potential of the gas particles and N is their total number in the container. Assuming again that $F = U - TS$, find an expression for μ in terms of a partial derivative of F ; make sure to specify what is held constant.

This question is done in the notes and the slides.

QUESTION A.8 QUALITATIVE PHYSICS: Be quantitative if you can - but the answers also require qualitative physical understanding.

(i): Why does the entropy of a pair of systems usually increase when one combines them physically into a single system? When does it not increase? If one then takes a single system, and separates it into two systems, the entropy also increases - why is this?

(ii): Suppose I have two sets of spin-1/2 systems, each containing N spins in an applied magnetic field which splits each spin level by an amount $2\Delta_o$. We assume that they are in every way identical except that one of the N -spin systems is at temperature $T = 0$, while the other is at temperature $T = \infty$.

Now, I combine the 2 sets of spins. What is the final energy of the combined system? And if I do the combination in an irreversible way, what do you think is the final temperature?

The entropy of a pair of systems is always greater than equal to that of the 2 systems before they are combined, by the 2nd law of thermodynamics. It will be greater if the process is irreversible, the same if the process is reversible. The same is true of we separate a single system into two - the 2nd law applies to any changes.

A set of spins at $T = 0$ has all the spins in the ground state. Suppose the energies of the individual spins are $\epsilon_{\pm} = \pm\Delta_o$. Then the ground state energy of N spins is $E_o = E(T = 0) = -N\Delta_o$. If the spins are at $T = \infty$ the population of up and down spins is the same - the energy of the system is then $E(T = \infty) = 0$.

If we combine the 2 sets of spins, then the energy must be the same - we now have a total energy $E = -N\Delta_o$ for a set of $2N$ spins, ie., an average energy for the j -th spin of $\epsilon_j = -\Delta_o/2$.

To find the temperature, we must assume that the combination process is irreversible, because otherwise the spins would not equilibrate to a new temperature. To find this temperature, we could simply guess that it is given roughly by $kT \sim \Delta_o$, since that is the energy where we expect things to cross over between the $T = 0$ and $T = \infty$ limiting results. This would be good enough for a qualitative answer.

NB: If we wanted to find it exactly, we could simply use the result that for a set of 2-level systems with splitting $2\Delta_o$, the energy is $E(T) = -\Delta_o \tanh(\beta\Delta_o)$ per spin. Since we know from above that this energy is in our case equal to $-\Delta_o/2$, we have $-\Delta_o \tanh(\beta\Delta_o) = -\Delta_o/2$, so that the answer is eventually

$$kT = \frac{\Delta_o}{\tanh^{-1}(\frac{1}{2})} \quad (0.1)$$

and we see that our guess that $kT \sim \Delta_o$ was a reasonably good one.

QUESTION A.9: RADIATION PRESSURE The radiation pressure p from photons is equal to $p = 4J/3c$, where J is the radiation flux. A star like the sun emits black-body radiation with flux $J = \sigma T^4$ per unit area of its surface, where temperature T is measured in Kelvin units; here $\sigma = 5.67 \times 10^{-8} W m^{-2} K^{-4}$, and the sun's surface temperature is $6,000 K$. The radius of the sun is $R_S \sim 0.7 \times 10^6$ km.

(i): Consider the forces on an electron at the sun's surface. If the cross-section for photon-electron scattering is $\sim 6.6 \times 10^{-29} m^2$, and the electron mass is $\sim 9 \times 10^{-31} kg$, then how do the gravitational and radiation forces on the electron at the sun's surface compare (assume here that all the photon energy is taken up by the electron)? You can assume that the solar mass is $2 \times 10^{30} kg$, and that the gravitational constant $G = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$.

NB: This question would certainly take longer than 20 mins - more like 35 mins

Here is what I find. The radiation flux at the sun's surface is $J(R_S) = \sigma T^4$, which I get to be $J(R_S) = 4.35 \times 10^7 W m^{-2}$. This gives a pressure $p(R_S) = 4J(R_S)/3c = 0.327 N m^{-2}$, using a value $c = 2.998 \times 10^8 ms^{-1}$. On an electron, with cross-section $6.6 \times 10^{-29} m^2$, this gives a force $f_p(R_S) \sim 2.157 \times 10^{-29} N$. The gravitational force is just $f_g(R_S) = -GM_S m_e / R_S^2$, which using the numbers given (and a mass $m_e = 9.11 \times 10^{-31} kg$ for the electron) the result $f_g(R_S) = -2.45 \times 10^{-28} N$. Thus in this calculation the total force $f(R_S) = f_p(R_S) + f_g(R_S)$ is almost entirely gravitational (in reality, charged particles are accelerated away from the sun by shock waves and coronal mass ejections).

(ii) How do the radiation force and gravitational force on the electron behave as a function of the distance r from the sun (for $r > R_o$)?. What then is the equation of motion for $r(t)$, and what is its solution as a function of time, if

the electron starts at a distance $r_o = r(t = 0)$ from the sun?

Both the radiation force and the gravitational force fall off like $1/r^2$. The radial equation of motion of the particle is then just $m\ddot{r} = -f_g(r)$, neglecting the radiation force, so that $\ddot{r}(t) = -GM_S/r^2$. Although this is not part of the course, the easiest way to solve this problem is to use the conservation of total energy for the electron. I did not specify the initial velocity, but suppose that at $r = r_o$, we have $\dot{r} = 0$. Then we have the result, by conservation of energy, that

$$\left(\frac{dr}{dt}\right)^2 = 2GM_S \left(\frac{1}{r(t)} - \frac{1}{r_o}\right) \quad \Rightarrow \quad t = \frac{1}{(2GM_S)^{1/2}} \int_{r_o}^{r(t)} dr \left[\frac{1}{r} - r_o^{-1}\right]^{-1/2}$$

and I don't expect anyone to do more than this.

QUESTION A.10 ENERGY FLUCTUATIONS: We start from the canonical partition function \mathcal{Z} for a system \mathcal{S} .

(i) Show that the “mean energy squared” of the system \mathcal{S} is given by $\langle E^2 \rangle = \mathcal{Z}^{-1}(\partial^2 \mathcal{Z} / \partial \beta^2)$.

(ii) Using this result work out an expression for the mean squared energy fluctuation in the energy of the system, written as $\langle \Delta E^2 \rangle = \langle (E^2) - \langle E \rangle^2$; and then show that it can be written as

$$\langle \Delta E^2 \rangle = -\frac{\partial \langle E \rangle}{\partial \beta}$$

these derivations are done in the notes and on the slides

END of ‘MOCK’ SHORT QUESTIONS