PHYS 403: MID-TERM TEST

(Feb 28th, 2020)

This IS A MID-TERM TEST. It will last 45 mins. The only material you will be allowed to use will be pens, pencils, and erasers. No notes of any kind are permitted, nor any calculators. You should try to do **all four questions**.

QUESTION (1) FREE ENERGY: For a gas of particles, the infinitesimal changes dS in the entropy and dV in the volume of the gas container result in a change dU in the energy, given by $dU = TdS - pdV$, where T is the temperature and p the pressure in the gas.

(i) The free energy of the system is $F = U - TS$. Find an expression for the infinitesimal dF, and show that the pressure is then given by the partial derivative $p = -(\partial F/\partial V)|_T$, where T is held constant.

(ii) Suppose we are allowed to add particles to the gas as well, so that $dU = TdS - pdV + \mu dN$, where μ is the chemical potential of the gas particles and N is their total number in the container. Assuming again that $F = U - TS$, find an expression for μ in terms of a partial derivative of F; make sure to specify what is held constant.

QUESTION (2) CANONICAL PARTITION FUNCTION: Suppose we have a system $\mathcal{S}$ which is immersed in a thermal bath $\mathcal{B}$ which is at temperature T, such that energy can pass between $\mathcal{S}$ and $\mathcal{B}$; but no matter can pass between them. If the system $\mathcal{S}$ has allowed energies E_j, then the **canonical** partition function is $\mathcal{Z}(\beta) = \sum_j e^{-\beta E_j}$, where $\beta = 1/k_B T$ is the inverse temperature.

(i) The mean energy of the system is given by $U = \sum_j p_j E_j$, where p_j is the probability that the state with energy E_j is occupied. Show that $U = \langle E \rangle = -\mathcal{Z}^{-1}(\partial \mathcal{Z}/\partial \beta)$.

(ii) Suppose the system $\mathcal{S}$ is a single 2-level system, with possible energies $E_\pm = \pm\Delta_o$. Write down an expression for the partition function $\mathcal{Z}$ for this system, and show that the mean energy at temperature T is given by $U = -\Delta_o \tanh \beta\Delta_o$.

QUESTION (3) ENERGY FLUCTUATIONS: We again start from the canonical partition function $\mathcal{Z}$ for a system $\mathcal{S}$.

(i) Show that the "mean energy squared" of the system $\mathcal{S}$ is given by $\langle E^2 \rangle = \mathcal{Z}^{-1}(\partial^2 \mathcal{Z}/\partial \beta^2)$.

(ii) Using this result work out an expression for the mean squared energy fluctuation in the energy of the system, written as $\langle \Delta E^2 \rangle = (\langle E^2 \rangle - \langle E \rangle^2)$; and then show that it can be written as

$$\langle \Delta E^2 \rangle \;\; = \;\; -\frac{\partial \langle E \rangle}{\partial \beta}$$

QUESTION (4) HARMONIC OSCILLATOR: Suppose we have a 1-dimensional oscillator with energy levels $\epsilon_n = (n + \frac{1}{2})\hbar\omega_o$, where ω_o is the frequency of the oscillator.

(i) Show that the partition function for this oscillator is $\mathcal{Z}(\beta) = \frac{1}{2} cosech\,(\hbar\beta\omega_o/2)$, where $cosech\, x \equiv 2/(e^x - e^{-x})$.

(ii) Using this result, and the result given in question 2, part (i) above, find an expression for the mean energy U of the system. Then draw a rough graph of this result for the energy U as a function of $x = k_B T/\hbar\omega_o$.

END of MID-TERM TEST