PHYS 403: FINAL EXAM 2021 – MARKING SCHEME (SECTION B)

Each question nominally counts for 30 points. In some cases there will be bonus marks so it is possible for somebody to get more than 30 for a question, and it is possible for their total mark to exceed 60, for the 2 questions they have to answer. Please be flexible in marking and give a bonus where deserved.

No need whatsoever to provide any details or explanations of the marks you award - but you should indicate what marks you give for each part of the questions you mark, and also, wherever there is any ambiguity or subtlety about how you award a mark, make some sort of note that will allow you, at some later date, to recall your reasoning for awarding the mark that you did give.

Please (i) compile an excel page that shown which mark you gave to each student for each question, along with a final total, for section B, for each student; and then (ii) add this mark to the main Excel page which has all the marks for the different homework assignments.

SECTION B: LONG QUESTIONS (ANSWER 2 of THESE)

QUESTION B.1: SUPERFLUID ^{4}He   Superfluid 4He is the best known neutral superfluid; this question looks at some of its properties.

(i): Draw the energy dispersion relation for quasiparticles in superfluid ^{4}He (ie., the plot of the energy $\epsilon_{\mathbf{p}}$ as a function of the momentum p). Now, explain why it is that an object with mass $M \gg m_4$ (where m_4 is the mass of a ^{4}He atom), which is moving through superfluid ^{4}He, will move without friction until it reaches a critical velocity $v_c \sim min(\epsilon_{\mathbf{p}}/|\mathbf{p}|)$. You should consider the problem at finite T, where thermally excited quasiparticles already exist.

This material is in the course notes. The ^{4}He dispersion relation graph should show the phonons, rotons, etc., and should show the energy and momentum scales, in appropriate units. The argument for the critical velocity needs to be done first as in the notes, by assuming the object has initial momentum $M\mathbf{V}$ and excites a quasiparticle with momentum $\hbar\mathbf{k}$ and energy $\epsilon_{\mathbf{k}}$, leaving the object with final momentum $M\mathbf{V}'$. By doing the elementary kinematics, and dropping the "recoil term" $\hbar^2 k^2/2M$, we then get the condition that $v_c \sim min(\epsilon_{\mathbf{p}}/|\mathbf{p}|)$, when $\mathbf{V}$ is parallel to $\mathbf{k}$ (see notes).

I am not sure how many of them will try to extend this argument to include scattering off thermally excited quasiparticles. The energy and momentum conservation equations are then (see course notes) $M\mathbf{V} + \hbar\mathbf{k} = M\mathbf{V}' + \hbar\mathbf{k}'$ and $\frac{1}{2}MV^2 + \epsilon_{\mathbf{k}} = \frac{1}{2}M(V')^2 + \epsilon_{\mathbf{k}'}$. If we again do the kinematics, and drop the recoil term (now equal to $\hbar^2(|bfk - \mathbf{k}'|^2/2M)$, we get the condition $v_c \sim min(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'}/\hbar|k - k'|)$, with $\mathbf{k}$ parallel to $\mathbf{k}'$.

I give **TOTAL mark = NINE (9) points**. This includes 1 for the drawing, 3 for the $T = 0$ explanation, and 5 for the finite T explanation.

(ii) Suppose that the object of mass M were to be moving in some fluid along the $\hat{x}$ direction. Suppose also that this fluid has a constant viscosity coefficient η, so that there is a force $-\eta\mathbf{v}(t)$ acting on the particle in the direction opposite to its velocity $v(t)$ along $\hat{x}$. Find the equation of motion of the particle, assuming that there is also an external force $f(t)$ acting on it along $\hat{x}$.

Then show that if (i) this force is $f(t)$, and (ii) the initial velocity at $t = 0$ is $v(t = 0) = v_o$, then the solution to the equation of motion is

$$v(t) \;=\; v_o e^{-\gamma t} \;+\; \int_0^t dt' \, \frac{f(t')}{m} \, e^{-\gamma(t-t')}$$

where $\gamma = \eta/M$.

Now, suppose that the force is actually a constant in time, so that $f(t) \to f_o$. Show that after a long time has elapsed, the particle will then reach a constant terminal velocity v_f, and give the result for v_f in terms of f_o, γ, and M. How could you have very simply derived this result without solving the equation of motion?

The derivation of the result is just a standard exercise in differential equations. The terminal velocity when $f(t) = f_o$ is obtained by going to long times (when $\gamma t \gg 1$). It is easily deduced by balancing forces, to be $v_f = f_o/\eta \equiv f_o/\gamma M$.

I give **TOTAL mark = EIGHT (8) points**. This includes 1 for the eqtn. of motion, 6 for deriving the solution, and 1 for the terminal velocity.

(iii): In a superfluid things are a little different because the friction depends on the velocity. Suppose that the friction coefficient in the superfluid behaves with velocity according to $\eta(v) = \eta_o(v - v_c)\,\theta(v - v_c)$, where $\theta(x)$ is just the Heaviside or "step" function (so $\theta(x) = 0$ for $x < 0$, and $\theta(x) = 1$ for $x > 0$).

To solve the equation of motion here is complicated - but you can still find the new terminal velocity v_f without doing this. Find the result for v_f.

The same arguments now give $v_f = v_c + f_o/\eta_o$.

I give **mark = TWO (2) points** for this.

(iv): In a real superfluid one can also have quantized vortex ring excitations, which behave like quasiparticles in that they can also be excited by interactions with an external body. For a circular ring, one has approximately that the energy E of the ring and the momentum p of the ring depend on the radius R of the ring according to

$$E \sim \tfrac{1}{2}\rho\kappa^2 R \ln\frac{R}{a_o} \; ; \qquad\qquad p \sim \pi\rho\kappa R^2$$

where ρ is the superfluid density, κ the circulation quantum, and $a_o \sim 0.1\ nm$ is a vortex core radius.

If in analogy with the quasiparticle argument, we suppose that the critical velocity for formation of a vortex ring is given by $v_c \sim min(E/p)$, then show that for a superfluid in which R can be as large as you like (a superfluid moving past an infinitely large object), then $v_c \to 0$. Show also that if the superfluid is moving through a cylindrical tube of radius R_o, then v_c is finite, and give an expression for it.

Finally, noting that the velocity of the vortex ring excitation itself is given by $v = dE/dp$, find an expression for the velocity $v(R)$ as a function of the vortex ring radius, and sketch a graph of it.

From the formulas for E and p we have

$$v_c \;=\; min(E/p) \;=\; min\left[\frac{\kappa}{2\pi R}\ln\frac{R}{a_o}\right]$$

and this is minimized when $R \to \infty$. In a cylinder of radius R_o, it is minimized when $R = R_o$, so that

$$v_c \;=\; \left[\frac{\kappa}{2\pi R_o}\ln\frac{R_o}{a_o}\right]$$

The velocity v is given by

$$v(R) \;=\; \frac{dE}{dR}\frac{dR}{dp} \;=\; \frac{\kappa}{4\pi R}\left[\ln\frac{R}{a_o} + 1\right]$$

using $dR/dp = (2\pi\rho\kappa R)^{-1}$. The sketch is easy.

I give **TOTAL mark = ELEVEN (11) points**. This includes 4 for the general result for the vortex critical velocity, 2 for the critical velocity in a tube, 4 for the result for $v(R)$, and 1 for the drawing.

QUESTION B.2: DIATOMIC GAS A diatomic molecule has 3 degrees of freedom, viz., translational motion of the molecular centre of mass, rotational motion abut the centre of mass, and vibrations in distance between the 2 atoms. We will treat these different degrees of freedom as being independent, ie., with no coupling between them. We assume the diatom is made from 2 atoms, each with mass m, and mean separation a_o.

(i): The moment of inertia of the rotating diatom is $I = 1/2ma_o^2$. We also suppose that the frequency of small harmonic oscillation of the distance x around the mean a_o between the atoms is ω_o.

Show that we can write the total canonical partition function $\mathcal{Z}$ for a gas of N such diatoms as $\mathcal{Z} = Z_{tr}Z_{rot}Z_{vib}$, where Z_{tr} comes from the translational degrees of freedom, where $Z_{rot} = z_I^N$ and $Z_{vib} = z_{\omega_o}^N$, and show that

$$z_I \;=\; \sum_{j=0}^{\infty}(2j+1)\exp[-\beta\hbar^2 j(j+1)/2I] \; ; \qquad\qquad z_{\omega_o} \;=\; \sum_{n=o}^{\infty}\exp[-\beta\hbar(n+\tfrac{1}{2})\omega_o]$$

You do not have to evaluate the translational term Z_{tr}.

The partition function is written as a product because the 3 degrees of freedom are independent. Both terms are in the usual form $z = \sum_i g_i \exp[-\beta E_i]$, where g_i is the degeneracy of the i-th state. For the rotator, $E_j = \hbar^2 j(j+1)/2I$ and $g_j j(j+1)$; for the oscillator, $E_n = \hbar(n+\frac{1}{2})\omega_o$, and $g_n = 1$.

This gets **TWO (2) marks**, one for each result.

(ii) Let us first consider the vibrational modes. Evaluate the partition function $z_{\omega_o}(\beta)$, and then show that the vibrational contribution to the energy of the system is $U_{vib}(\beta) = \frac{1}{2}N\hbar\omega_o \coth(\beta\hbar\omega_o/2)$. From this find also the contribution $C_V^{vib}(\beta)$ to the specific heat.

Finally, sketch the behaviour of both $U_{vib}(\beta)$ and $C_V^{vib}(\beta)$ as functions of the temperature T.

To get $z_{\omega_o}(\beta)$, $U_{vib}(\beta)$, and $C_V^{vib}(\beta)$ are standard exercises for an oscillator. One gets, by summing the series for $z_{\omega_o}(\beta)$, and then doing the appropriate differentiations, that

$$z_{\omega_o}(\beta) \;=\; \tfrac{1}{2} cosech\left(\frac{\hbar\omega_o}{2k_BT}\right) \;; \qquad U_{vib}(\beta) \;=\; \tfrac{1}{2}\hbar\omega_o \coth\left(\frac{\hbar\omega_o}{2k_BT}\right) \;; \qquad C_V(T) \;=\; \left(\frac{\hbar\omega_o}{2k_BT}\right)^2 k_B\, cosech^2\left(\frac{\hbar\omega_o}{2k_BT}\right)$$

The sketches of the behaviour are also standard.

For the derivation of the 3 formulae, give a total of **FIVE (5) marks**. For the sketches give a total of **THREE (3) marks**. If you want to slightly alter the weighting go ahead, but the **TOTAL** should add up to **EIGHT (8) marks**.

(iii) Now let's look at Z_{rot} for the rotational motion of the diatom. The low T behaviour is easy, because the terms in the sum in the expression for $z_I(\beta)$ decrease rapidly with increasing j. By taking just the first 2 terms in the sum, find a simple low-T result for $z_I(\beta)$, and from this find expressions for $U_{rot}(T)$ and $C_V^{rot}(T)$ for the N diatoms in the low T regime.

For the high-T behaviour we need to approximate the sum as an integral. Using the result $\int_0^\infty dx\, x\, e^{-x^2} = 1/2$, find a simple result for $z_I(\beta)$ in the high-T regime where $kT \gg \hbar^2/2I$, with the result $\propto kT$. Then, from this result, find the energy U_{rot} and $C_V^{rot}(T)$ for the N diatoms in the high T regime.

Finally, plot sketches for U_{rot} and $C_V^{rot}(T)$ for the N diatoms as a function of T; you can use the expression you found for the low-T and high-T results, and then just simply interpolate between them.

The low T behaviour is defined by $\beta E_j \gg 1$ for all j except $j=0$, where $E_j = \hbar^2 j(j+1)/2I$ as we found above. Physically this means that $k_BT \ll$ than the gap $2\bar{\Delta} = \hbar^2/I$ between the ground state and the first excited state, so that $\beta\bar{\Delta} \gg 1$. In this case we can ignore all terms in $z_I(\beta)$ except for the first two (ie., $j=0$ and $j=1$), to get

$$z_I(\beta) \;\to\; 1 + 3e^{-\beta\hbar^2/I} \;\;\equiv\;\; 1 + 3e^{-2\bar{\Delta}/k_BT} \qquad (\frac{k_B}{\bar{\Delta}} \to 0)$$

We then find, for N particles, that

$$U_{rot} \;=\; -(\partial \ln Z/\partial\beta) \;=\; \frac{3N\hbar^2}{I} e^{-\beta\hbar^2/I} \;; \qquad C_V^{rot}(T) \;=\; -k_B\beta^2 \frac{\partial U}{\partial\beta} \;=\; 3Nk_B\left(\frac{\hbar^2\beta}{I}\right)^2 e^{-\hbar^2\beta/I} \qquad (\frac{k_B}{\bar{\Delta}} \to 0)$$

The high T regime obtains when the spacing between at least the lowest levels $\ll k_BT$, ie., when $k_BT \gg \bar{\Delta}$, with $\bar{\Delta} = \hbar^2/I$ as before, ie., when $\beta\bar{\Delta} \ll 1$. To convert the sum to an integral, we write $x = j(j+1)$, so that $dx = 2j+1$. It then follows that if the function $f(j(j+1)) \equiv f(x)$ varies slowly with x (ie., that $df/dx \ll 1$, then $\sum_j (2j+1) f(j(j+1)) \equiv \int dx f(x)$. Thus we have

$$z_I(\beta) \;=\; \sum_{j=0}^{\infty} j(j+1)\, e^{-\beta\bar{\Delta}\, j(j+1)} \;\;\to\;\; \int_0^\infty dx\, e^{-\beta\bar{\Delta}x} \;=\; \frac{k_BT}{\bar{\Delta}} \;\equiv\; \frac{2Ik_BT}{\hbar^2} \qquad (\frac{k_B}{\bar{\Delta}} \to \infty)$$

Then, for N particles, it immediately follows that

$$U_{rot} \;=\; Nk_BT \;; \qquad C_V^{rot}(T) \;=\; Nk_B \qquad (\frac{k_B}{\bar{\Delta}} \to \infty)$$

Note that in the question I gave them the integral $\int_0^\infty dx\, x\, e^{-x^2} = 1/2$, which would be used if I had made the substitution $x^2 = j(j+1)$ above, nut of course it comes to the same thing.

To draw graphs, we note that the energy and specific heat are exponentially small for low T, with energy $\propto T$, and specific heat $\sim$ constant, for high T, with the crossover at $k_B \sim \bar{\Delta}$.

The TOTAL mark here will be **SIXTEEN (16) MARKS**.

For the low T results, give a total of **SIX (6) marks**, ie., 2 each for the partition function, the energy, and the specific heat.

For the high-T results, give a total of **TEN (10) marks**, with 8 for the derivation using the integral, and 2 for the results for U and C_V.

(iv) The "third" contribution to the specific heat coming from the translational degrees of freedom is just that from a 2-dimensional classical Maxwell-Boltzmann gas. Typically, the vibrational zero point energy $\hbar\omega_o/2 \gg \hbar E_o$, where $E_o = \hbar^2/2I$ is the rotational zero point energy. Using the results you have derived above for $C_V^{rot}(T)$ and $C_V^{vib}(T)$, sketch the result you expect for the TOTAL specific heat $C_V(T)$ for a gas of N diatoms, as a function of T. Explain the limiting behaviour you find for $C_V(T)$ for (i) high T (ie., for $T \gg \hbar\omega_o/2$) and for low T (ie., for $kT \ll \hbar^2/2I$)?

There are 3 temperature regimes, viz., (a) the low-T regime, when $T < T_1 = \hbar^2/2Ik_B$; (b) the intermediate regime, when $T_1 < T < T_2 = \hbar\omega_o/2k_B$, and (c) the high-$T$ regime, when $T > T_2$. Well below $T < T_1$ we have the free Maxwell-Boltzmann gas result that $C_V(T) = 3Nk_B/2$. In the intermediate regime we have the extra rotational contribution, which flattens out at Nk_B, so that now $C_V(T) \sim 5Nk_B/2$. Finally, in the high-T regime, we get the oscillator contribution which rapidly flattens out to Nk_B, so that we get $C_V(T) \sim 7Nk_B/2$. So a graph would show these flat regions with a smooth crossover between them.

For the correct sketch, and a proper understanding of how to get it, give **FOUR** (4) marks.

GRAND TOTAL = 30 MARKS

QUESTION B.3: The YOUNG and the OLD UNIVERSE Near the beginning its life, the universe was composed of a variety of fermionic particles, plus photons. Near the end of its life (using extrapolations from what we already know), it will be a mixture of black holes and photons.

(i) Describe the universe as it was until a time $t = \tau_o$ after the Big Bang, where $\tau_o \sim 400,000$ yrs (you can ignore the time in the first few years after the Big Bang). What happened around $t \sim \tau_o$, and why? Why did this happen when the temperature $T \sim 4,000$K?

During the time period in question, the universe was an expanding hot plasma composed largely of electrons, positrons, neutrinos, some heavier nucleons, and photons. The photons scattered strongly off charges, so the photons were in dynamic thermal equilibrium with the matter.

However, as the temperature fell, the charged articles began to combine into neutral objects. After the electrons and positrons combined, all that was left was for nucleons to find electrons to form atoms. The key process as the combination of protons and electrons to form neutral H. This happened at a temperature $\sim 4,000$ K, much lower than the ionization energy of H, simply because the gas had a rather low density. **NB**: the students need to explain why a low-density gas will ionize at a much lower temperature than the ionization energy).

The recombination suddenly made the universe transparent to photons, which have been traveling ever since, and constitute the "microwave background".

For this section give **SIX (6) marks**, with 2 for the explanation of why the charged plasma is opaque and the neutral plasma is transparent, and 4 for explaining why recombination happens at 4,000K. If any students try to give a quantitative explanation of why 4,000K, and give a good argument, they should get bonus marks.

(ii) In the earlier stages of the universe (for $t \ll \tau_o$), we can assume that the system is ultra-relativistic, meaning that the fermion particle energy $\epsilon \gg mc^2$, where m is the fermion rest mass. We can also assume the system is at very high temperature, so that $|\mu|/kT \to 0$. Under these conditions, show that the energy of a fermion with momentum p is $\epsilon_p \sim pc$, and find expressions for (a) the number density $\rho = N/V$, and (b) the energy density $u = U/V$, for the fermions - showing in particular that $u \propto T^4$. You can write the answers in terms of the definite integrals

$$F_n = \int_0^\infty dx \frac{x^n}{e^x + 1}$$

which you do **not** need to evaluate.

To show that $\epsilon_p = pc$, we simply note that a relativistic particle with rest mass m has energy dispersion $E^2 = p^2c^2 + m^2c^4$. Taking the square root of this and expanding for $pc \gg mc^2$, we get $E = [pc + \frac{1}{2}m^2c^3/p + \cdots]$, proving the result.

To find the number density $\rho = N/V$, we simply write

$$\rho(T) \;=\; \int \frac{d^3p}{(2\pi\hbar)^3} f(\epsilon_p) \;=\; \frac{4\pi}{(2\pi\hbar)^3}\int_0^\infty dp \frac{p^2}{e^{\beta pc}+1} \qquad \rightarrow \qquad \frac{1}{2\pi^2}\left(\frac{k_BT}{\hbar c}\right)^3 F_2$$

and for the energy density, by the same manouevres, we get

$$u(T) \;=\; \frac{1}{2\pi^2}\frac{(k_BT)^4}{(\hbar c)^3} F_3$$

where the integrals F_2, F_3 were already defined.

Give a total of **EIGHT (8) marks** here; 2 for the explanation of why $\epsilon_p = pc$, and 6 for the derivations of $\rho(T)$ and $u(T)$.

(iii) In the universe at present (at a time $t \sim 1.4 \times 10^{10}$ yrs after the Big Bang), the universe is populated by a mixture of matter and radiation (plus the enigmatic "dark matter"). The matter is a mixture of stars and black holes, along with a lot of sub-stellar "junk" (sub-stellar brown dwarfs, planets, planetoids, dust, gas, etc.).

Most of the stars will end up as black dwarfs (ie., cold white dwarfs) after a time period extending up to $\sim 10^{14}$ yrs; how does this come about? Which stars will not end up as black dwarfs, and what will happen to them?

Low-mass black dwarfs can be treated using the usual non-relativistic Chandrasekhar argument, to show that their radius $R_o = cM^{-1/3}$, where the constant $c = 2C_f/GC_g$ is derived by minimizing the sum of the degeneracy energy $U_f = C_f M^{5/3}/R^2$ and the gravitational energy $U_g = -GC_gM^2/R$, where R is the black dwarf radius, and M is its mass. Suppose now that the dwarf has not yet cooled (ie., it is still white), so that there is an extra small radiative thermal contribution $U_T = \alpha(T,M)R^2$ to the energy. Assuming that $\alpha(T,M)$ is "small", find the new solution $\tilde{R} = R_o + \delta R$ for the radius, by looking for the small correction to the original minimization equation (formally, we assume that both δR and α are $\sim O(\epsilon)$, where $\epsilon \ll 1$, and isolate terms $\sim O(\epsilon)$ in our equations).

Stars with initial mass $\lesssim 8$ solar masses will live out their life on the main sequence, followed by a red giant phase which then leads to a white dwarf. During the main sequence phase they burn up H, He, etc., up to a point where the internal temperature can no longer lead to fusion with higher-mass nucleons. The increase in luminosity as the heavier nucleons are burnt at higher T, along with core shrinkage required to support the higher T, leads to a blowing out of the outer layers to the red giant, and much mass loss. The remaining core shrinks to the white dwarf, of density $\sim 10^6\ g/cm^3$. This white dwarf very slowly cools. The very small fraction of stars with initial mass $\gtrsim 8$ solar masses will continue to synthesize ever heavier nucleons by fusion, with massive rise in core temperature and shrinkage in core volume, until Fe is synthesized. At this point fusion is no longer possible because heavier nucleons have shrinking mass deficit, and so the star collapses very suddenly - subsequently one has a supernova and either a neutron star or black hole remnant.

If the white dwarf is not radiating, then we have the usual balance between electron degeneracy and gravitational pressure, found by minimizing the energy $U = U_e + U_G = C_e(M^{5/3}/R^2) - GC_G(M^2/R)$ to give

$$GC_G\frac{M^2}{R^2} - 2C_e\frac{M^{5/3}}{R^3} \;=\; 0 \qquad \Longrightarrow \qquad R \rightarrow R_o \;=\; \frac{2C_e}{GC_G}M^{-1/3}$$

The extra thermal contribution to the energy leads to a radiation pressure; we suppose it is small relative to U_e and U_G, as will be the correction δR to R_o. Let's write $alpha(T,M) = \epsilon A(T,M)$ and $\delta R = \epsilon \Delta R$, where $\epsilon \ll 1$. Then our new minimization equation becomes (after multiplying throughout by R^3) that

$$GC_GM^2R - 2C_eM^{5/3} + \epsilon AR^4 = 0 \qquad \Longrightarrow \qquad GC_GM^2(R_o + \epsilon\Delta R) - 2C_eM^{5/3} + \epsilon A(R_o^4 + 4\epsilon R_o^3) = 0$$

If we now equate the terms in ϵ, and ignore the term $\sim O(\epsilon^2$, this then leads (noting that $GC_GM^2R_o - 2C_eM^{5/3} = 0$) to the result that

$$\delta R \;=\; -\frac{\alpha}{GC_GM^2}\left(\frac{2C_e}{GC_G}M^{-1/3}\right)^4 \;=\; -16\alpha(T,M)\frac{C_e^4}{G^5C_G^5}M^{-10/3}$$

Give a total of **TWELVE (12) marks** here; this is 4 for a correct discussion of the history leading to white dwarves, black holes, etc., and 8 for the derivation of the result for δR.

(iv) After extremely long times almost all matter will amalgamate into black holes, apart from a photon bath which steadily cools (after far longer times $> 10^{100}$ yrs, almost all of the black holes will decay by the Hawking process into radiation as well). Suppose at some given time the volume of the universe is V_H. Using the Planck result that the total photon energy of the universe $U_{ph} \propto V_H T^4$, show that the photon bath obeys $C_V(T) \propto V_H T^3$, and from this, assuming that the expansion of the universe is adiabatic, find the dependence of the entropy on V_H and T.

Finally, let us assume that the expansion of the universe obeys the Hubble law, so that $V_H(t) \propto t^3$, and using the Planck result that the energy density of a photon gas is given in MKS units by

$$u(T) \;=\; \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3}$$

find (a) the temperature of the photons after a time $t = 700\times 10^9$ yrs, assuming that at the present time of $t = 1.4\times 10^{10}$ yrs, one has $T = 2.7\ K$; and (b) find the photon energy density of the universe at the present time, in MKS units.

The photons form an isolated system after decoupling from matter, so their expansion is adiabatic. Since $U(T) \propto V_H T^4$, and $TdS = dU + pdV$, we can write

$$C_V(T) \;=\; T\left(\frac{\partial S}{\partial T}\right)_V \;=\; \left(\frac{\partial U}{\partial T}\right)_V \propto V_H T^3 \qquad \Longrightarrow \qquad S \propto V_H T^3 \;=\; constant$$

We have $T \propto (S/V_H)^{1/3} \propto 1/t$. Thus the temperature $T(t = 7\times 10^{11}) = 2.7/50 = 54\ mK$.
At the present time where $t = 1.4\times 10^{10}$ yrs, where $T = 2.7\ K$, we have

$$u(T) \;=\; \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3} \quad\Longrightarrow\quad u(T) \;\sim\; 4.01\times 10^{-14}\ Jm^{-3}$$

Give a total of **FOUR (4) marks**, viz., 2 for teh derivation of $C_V(T)$, and 2 for the 2 correct numbers.
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GRAND TOTAL = 30 MARKS

QUESTION B.4: METALS and INSULATORS Solids can be classified into metals or insulators. Very roughly speaking, we can say that (a) Metals have mobile electrons, with dispersion relation $\epsilon_p \sim p^2/2m_1$, a Fermi surface, and behave similarly to an electron gas, whereas (b) electrons in an insulator with energy near the Fermi energy are bound to atoms and not mobile, and have no Fermi surface. Only above an "energy gap" are they mobile, and we can write an approximate dispersion relation $\epsilon_p \sim E_o + p^2/2m_2$. The "masses" m_1 and m_2 are not necessarily equal to the free electron mass m_o. Typically the gap $E_o \sim 1-2\ eV$ in size.

(i) Noting we also have acoustic and possibly optical phonons, draw pictures of how you think the specific heat $C_V(T)$ will behave as a function of temperature T, for both metals and insulators, and explain why the different contributions have the temperature dependence that they do.

For the metal we have T and T^3 terms at low T, with the phonon T^3 contribution flattening off above the Debye temperature T_D. For the insulator we will have the T^3 term, plus a term like $e^{-E_o/kT}$.

Total is **SIX (6) marks**, with 2 for the T and T^3 terms, 2 for the exponential term, and 2 for the flattening off.

(ii) At low T, a degenerate fermion system shows a specific heat of form $C_V(T) \propto g(E_F)T$, where $g(E_F)$ is the 1-particle density of states at the Fermi energy. From this result, deduce the low-T behaviour of (a) the energy $U(T)$ (b) the entropy $S(T)$, and (c) the free energy $F(T)$. Can you give a qualitative argument which justifies the result you get for $U(T)$?

A useful way to measure the density of states $g(E)$ in a metallic system is to look at the rate of photon absorption by the metal as a function of photon frequency ω. Photons will only be absorbed if an electron can be excited from an occupied state at one energy to an unoccupied state at another higher energy. Draw what you think you would see for the photon absorption as a function of frequency ω in (a) a low T metal, and (b) a low T insulator, with $E_o = 2\ eV$.

For the form of these functions: We expect $U(T) \sim U_o + g(E_F)T^2$, just by integrating $C_V(T) = dU/dT$. Then, from $C_V(T) = T(dS/dT)$, we get $S = \int dT(C_V/T) \sim g(E_F)T$, and $F(T) \sim g(E_F)T^2$, like $U(T)$. Qualitatively, we have a fraction kT/E_F of fermions excited to a characteristic energy $\sim kT/2$.

The photon absorption should look like the density of states, ie., constant in E for metals, and like $[E/(E^2 - E_o^2)^{1/2}]\,\theta(E - E_o)$ for a semiconductor (provided we ignore the energy dependence of the matrix elements).

A total of **EIGHT (8) marks**, with 4 for the thermodynamic quantities, and 2 for their qualitative explanation, and 3 for the optical spectrum.

(iii) A very common approximation when dealing with acoustic phonons is to assume a phonon density of states $g(E) = 9E^2/(k_B T_D)^3$ for $0 < E < \theta_D$, and $g(E) = 0$ for $E > \theta_D$. Here T_D is the "Debye temperature" and $\theta_D = k_B T_D$ is the "Debye energy". Typically T_D is somewhere in the range $100\ K - 600\ K$ for different solids.

From this information, you should be able to derive an integral expression for $\ln \Xi(T)$ for the acoustic phonon system (where Ξ is the grand canonical partition function), and also for the energy $U(T)$ [HINT: use the analogy with photons]. Assume a system of unit volume, so that $U(T) = u(T)$, the energy density; and assume that the atoms taking part in acoustic vibrations each have mass M. You do not need to evaluate the integrals over energy. You will use the result that the phonon chemical potential $\mu = 0$; why is this the case?

Finally, we want to evaluate the root mean square displacement of atoms in the solid caused by acoustic phonons. This can be shown to be given by $\bar{x} = [\langle x^2 \rangle]^{1/2}$, where

$$\langle x^2 \rangle \;=\; \frac{\hbar^2}{2M} \int \frac{dE}{E} g(E)[1 + 2n(E)]$$

in which $n(E)$ is the Bose distribution function. Derive an integral expression for $\langle x^2 \rangle$, and then show that in the low temperature limit $T \ll T_D$, we have a finite $\bar{x}$ given by

$$\bar{x} \;\sim\; \tfrac{3}{2}\hbar(1/M\theta_D)^{1/2} \qquad\qquad (T \to 0)$$

How do you interpret this result physically?

The phonon chemical potential $\mu = 0$ for the same reason as for photons - because their number is not conserved. The grand partition function Ξ is given for a system of unit volume by

$$\ln \Xi \;=\; \int_0^\infty dE\, g(E)\, \ln(1 - e^{-\beta\hbar E}) \qquad \rightarrow \qquad \frac{9}{(k_B T_D)^3} \int_0^\infty dE\, E^2\, \ln(1 - e^{-\beta\hbar E})$$

and the energy density is then

$$u(T) \;=\; \frac{9h}{(k_B T_D)^3} \int_0^\infty dE \frac{E^3}{e^{\beta\hbar E} - 1}$$

If we now go to the displacement, we have

$$\langle x^2 \rangle \;=\; \frac{\hbar^2}{2M} \int \frac{dE}{E}\, g(E)\, [1 + 2n(E)] \qquad \rightarrow \qquad \frac{9\hbar^2}{M} \left[\frac{1}{4\theta_D} + \frac{1}{\theta_D^3} \int_0^{\theta_D} dE \frac{E}{e^{\beta\hbar E} - 1} \right]$$

which for low T (for $T \ll T_D$) gives

$$\langle x^2 \rangle \;\sim\; \frac{9\hbar^2}{4M\theta_D} \qquad \Longrightarrow \qquad \bar{x} \;\sim\; \tfrac{3}{2}\hbar(1/M\theta_D)^{1/2}$$

which just describes the zero-point motion of the system.

A total of **TEN (10) marks**, with 1 for μ explanation, 1 for $\ln \Xi$, 2 for $u(T)$, 2 for $\langle x^2 \rangle$, 2 for the limiting value of $\bar{x}$, and 2 for the ZPE explanation.

(iv) All of the above ignores the fact that in any real solid there will be defects (which behave like 2-level systems), electronic spin impurities, and nuclear spins. To isolate out the effect of electronic spin impurities in an insulator, we can apply a magnetic field. Suppose these impurities have spin-1/2, and we apply a magnetic field B to the system. What then is the partition function for a set of N such impurities, and what is their contribution to the specific heat? Finally, draw a graph of the resulting specific heat for an insulator in the range $0 < T < 50\ K$, assuming that (a) the

Debye temperature $T_D = 500\ K$, and (b) the magnetic moment of the spin impurities is $\mu/k_B = 0.7\ K/T$, where T means "Tesla", and we are in an applied field of $20\ T$.

The new contribution to the partition function will be $\mathcal{Z} = Z_1^N$, where $Z_1 = [\exp(\beta\mu B) + \exp(-\beta\mu B)]$, where here B is the applied field and μ the spin moment. The specific heat is then

$$C_V(T) \ = \ Nk_B(\beta\mu B)^2\, sech^2(\beta\mu B)$$

For a system in a field of $B = 20\ T$, so that $\mu B = 14\ K$, the impurity contribution to the specific heat peaks at $T \sim 6\ K$. How this compares with the phonon specific heat depends on N; I should have specified N for this question, and asked them to work out the phonon specific heat as well.

A total of **SIX (6) marks**, with 1 for $\mathcal{Z}$, 3 for the derivation of $C_V(T)$ (and 1 if it is not derived), plus 2 for the graph.

grand total is THIRTY (30) marks.

END of FINAL EXAM