PHYS 350: MOCK EXAM

(Dec 1st, 2004)

This is a mock exam for Phys 350. It will be just like the real exam, which will last 2 hrs and 30 mins. You should try it out under simulated conditions of a real exam, once you think you are ready.

This exam will last 2 hrs and 30 mins. The only material allowed into the exam will be pens, pencils, and erasers. No notes of any kind are permitted, nor any calculators.

There are 2 sections. Students should answer THREE QUESTIONS ONLY from section A, and TWO QUESTIONS ONLY from section B. No extra marks will be given for extra questions answered. The questions in section A should take roughly 15-20 minutes to answer, and the questions in section B roughly 45-50 minutes to answer.

SECTION A

A1: A bicycle wheel of mass M and radius R rolls straight down a flat inclined plane (ie., the wheel axis stays horizontal, and the wheel rolls down the path of steepest descent without changing direction). The rolling happens without friction and without slipping.

(i) If we assume all the wheel mass is concentrated at the rim, at a distance R from the centre, calculate the moment of inertia of the wheel about its axis.

(ii) The plane is inclined at an angle of 45^o. If its height is 10m, and the wheel starts at the top of the plane at rest, then what will be its translational velocity when it leaves the bottom of the plane and begins to roll horizontally? Assume the gravitational constant is $g = 10ms^{-2}$.

A2: The energy of a system is defined as

$$E \;=\; \dot{\mathbf{q}} \cdot \frac{\partial L}{\partial \dot{\mathbf{q}}} - L \tag{1}$$

where $L(\mathbf{q}, \dot{\mathbf{q}})$ is the Lagrangian of the system, and the N-dimensional vectors $\mathbf{q}, \dot{\mathbf{q}}$ represent the generalised 'position' and 'velocity' of the system.

(i) If L does not depend explicitly on t, show, with the aid of Lagrange's equations, that $dE/dt = 0$, ie., that E is a constant of the motion.

(ii) Suppose we have a system with Lagrangian, in the centre of mass frame of the system, of the form

$$L(\phi, \mathbf{\Omega}) \;=\; \frac{1}{2} I_{\alpha\beta} \Omega_\alpha \Omega_\beta + V(\phi) - g\phi \cdot \mathbf{\Omega} \tag{2}$$

where ϕ is the angular orientation of the system, $\mathbf{\Omega} = \dot{\phi}$ is the angular velocity vector, and the last term in the Lagrangian is sometimes called a "spin-orbit" coupling term. Note that the summation convention is used in this expression for the inertial term, in terms of the components Ω_α, etc. (so that we could also write the term $g\phi \cdot \mathbf{\Omega}$ as $g\phi_\alpha \Omega_\alpha$).

What is Lagrange's equation for this system? (you can either give this as a vector equation or in terms of the components).

A3: A solid cylinder of length Z_o, radius R_o, and uniform density ρ, rolls without slipping and with negligible friction on a flat horizontal surface between 2 vertical parallel walls- its motion is such that it never comes near to either wall. The cylinder axis is always parallel to the walls. Each wall exerts an *attractive force* on the cylinder. The attractive force from the j-th wall on the cylinder is equal to kx_j, where k is a constant and x_j the distance of the cylinder axis from that j-th wall (and $j = 1, 2$ labels which of the 2 walls we refer to).

(i) Find the Lagrangian for the cylinder

(ii) Find the frequency of small oscillations of the cylinder

A4: A particle of mass m is in orbit about the earth, which has mass M; we assume a gravitational constant G.

(i) Starting from the Lagrangian for the particle, show that its radial coordinate r satisfies the equation of motion

$$m\ddot{r} - \frac{L^2}{mr^3} + \frac{GMm}{r^2} = 0 \tag{3}$$

and then use this equation to derive the orbital period (the time required to accomplish a single revolution around the earth) if the particle is in a circular orbit of radius R.

(ii) Suppose we now suddenly accelerate the particle up to escape velocity, by applying a thrust parallel to its motion. What will be the shape of the new orbit, and what will be the particle velocity immediately after the thrust is applied?

A5: A particle moves in a 1-dimensional potential having the "Morse" form, ie.,

$$V(x) = \frac{A}{x^{12}} - 2\frac{B}{x^6} \tag{4}$$

where $x > 0$ is the displacement of the particle form the origin, and $A, B > 0$. Now suppose the particle has very little kinetic energy, so that it stays very close to the bottom of the potential well. Show it then behaves like a harmonic oscillator, and find its frequency of oscillation.

SECTION B

B1: A spaceship of mass m approaches a mysterious massive object of mass M, under the influence of a potential $V(r) = -GMm/r^4$, where r is the distance between them. We specify the "initial state" of the spaceship, at a large distance from the object, by its initial free velocity $\mathbf{V}_o$ having initial "impact parameter" b. The impact parameter is defined as the distance of closest approach of the spaceship to the object in the *absence* of any interaction between them, given the initial velocity $\mathbf{V}_o$.

(i) Find the radial equation of motion for the spaceship, and classify its possible orbits. For a given value of b, one can now derive a "trapping criterion" for the spaceship, as a function of $V_o = |V_o|$; for what range of values of V_o will the spacecraft be sucked into the mystery object?

(ii) The spaceship wishes to rendezvous with a space station which is in circular orbit about the mystery object, with orbital period T. This orbit is beginning to go unstable. Express first the distance R_o of the space-station from the central mystery object, in terms of T. Then assume that the approaching spaceship decides to make a course correction to change its impact parameter (without changing its speed V_o), while still far from the central object, so that its minimum distance of approach to the object is also R_o. What must be the impact parameter, expressed in terms of V_o and T, so that it achieves this goal?

(iii) Draw a sketch of the motion of the spaceship that you have derived in (ii) (including the radial and angular motion).

B2: Two parallel rigid walls face each other, and to each of them is attached by springs a flat plate of mass M, so that each plate faces each other and has a natural oscillation frequency $\Omega_o = (k/M)^{1/2}$. However the oscillations of the 2 plates are coupled to each other via the air between them, so that there is an effective interaction between them of the form

$$V(x_1, x_2) = \alpha(\dot{x}_1 - \dot{x}_2)^2 \tag{5}$$

where $\alpha > 0$, and $\dot{x}_1$ and $\dot{x}_2$ are the horizontal velocities of the 2 plates.

(i) Using suitable coordinates, find the Lagrangian and equations of motion for this system.

(ii) What are the eigenfrequencies and normal modes of the system?

(iii) Suppose that at time $t = 0$ the system starts with the configuration $\dot{x}_1(t = 0) = A$, and $\dot{x}_2(t = 0) = 0$, and both are initially at their equilibrium positions. What is the subsequent motion of the plates?

(iv) Suppose we now add friction to the system. By physical argument, explain briefly where you think such friction might come from, and how it would enter the equations of motion (*not* the Lagrangian). Without solving these equations, explain how the friction would affect the motion, depending on how strong it was.

B3: Consider a rigid body with no particular symmetry, having 3 different principal moments of inertia, defined so that $I_1 < I_2 < I_3$.

(i) If the total rotational energy of the body is T, and the angular momentum has magnitude L, then show that

$$2TI_3 > L^2 > 2TI_1 \tag{6}$$

This inequality can be interpreted geometrically in terms of the Binet construction- how is this done?

(ii) The equation of motion for the angular momentum vector $\mathbf{L}$ of a rigid body is, *in the frame of the rigid body*, given by

$$\dot{\mathbf{L}} + (\mathbf{\Omega} \times \mathbf{L}) \ = \ \mathbf{N} \tag{7}$$

where $\mathbf{\Omega}$ is the angular velocity describing the rigid body rotation, $\mathbf{N}$ is the applied torque vector, and all three vectors are functions of time t. Eliminate $\mathbf{L}(t)$ to write down the Euler equations of motion for each component of $\mathbf{\Omega}(t)$ in this frame, and briefly explain why neither $\mathbf{L}$ nor $\mathbf{\Omega}$ will remain constant in general unless the body reduces to a spherical top (with $I_1 = I_2 = I_3$).

(iii) Now suppose we have a symmetrical top, with $I_1 = I_2$. Suppose that for $t < 0$, the system is in the simple state with $\mathbf{\Omega} = \Omega_o \hat{\mathbf{x}}_3$, independent of time. Then at $t = 0$ we apply a sudden torque $\mathbf{N}(t) = \hat{\mathbf{x}}_1 \delta(t)$. Now find the subsequent motion of the body (for $t > 0$).

(iv) For the symmetric top just discussed, what is the motion of $\mathbf{L}(t)$ and $\mathbf{\Omega}(t)$ in the *inertial frame* of the body centre of mass?

END of MOCK EXAM