REMARKS on GREEK MATHEMATICS

The history of Greek mathematics spans the period from Thales, around 600 BC, to the end of the 2nd Alexandrian school, around 415 AD. Enormous advances were made from 600-200 BC. The most notable early achievements were from the school associated with Pythagoras in c. 550 AD. Then some 250 yrs later (essentially in the period 300-200 BC), came the amazing achievements of the '1st Alexandrian school'. This school was associated with the great library in Alexandria (the most important of the many cities founded by Alexander the Great, this one at at the mouth of the river Nile - the library was put in place by Ptolemy, the successor to Alexander in Alexandria). This entire period is marked by an extraordinary creative flowering, nourished by several different mathematical schools- over this time period these led to huge advances. The 3 best known of the remarkable mathematicians of the 1st Alexandrian school were Archimedes, Euclid, and Apollonius; the first two of these, along with Pythagoras, are amongst the greatest mathematicians in history. Much of their work, along with that of Eratosthenes, Aristarchus, Hero, and Hipparchus, was lost during the early middle ages, and some of it was not to be surpassed until the 19th century.

The early work of the Pythagorean school had a very large effect on philosophy- it is unfortunate that the ideas of Archimedes came too late to have much influence, since they opened the door to quantitative mathematical physics. However by this time the great Greek intellectual outpouring was in decline, and never again acquired its first lustre. The Greek world was to be overtaken eventually by a very different kind of regime, that of the Roman empire. The Romans had little use for Greek mathematics- the first Roman to translate Euclid into Latin was Boethius, in 480 AD, some 800 years after Euclid had written his famous "*Elements*". One can hardly refrain from noticing the symbolic import when a Roman soldier killed Archimedes- in retrospect this spelt the end of the Greek enlightenment.

Nevertheless there was a second flowering of Alexandrian mathematics, beginning in roughly 30 BC; although not encouraged by the Romans, it was also left unhindered. The best-known mathematicians of the 2nd Alexandrian school were the remarkable Diophantus, who invented the branch of number theory now bearing his name, plus Pappus and Hypatia (daughter of Theon, himself also a member of this school). The 2nd Alexandrian school is also notable for the very influential astronomical work of Ptolemy. Mathematics at this later date was immersed in a rather different philosophical climate, that of the neo-Platonists and neo-Pythagoreans, whose effect on mathematical research was not terribly constructive. The 2nd Alexandrian school was brought to a sad close with the rise of the Eastern Christian church, which was very hostile to any kind of learning - with the murder of Hypatia by the Christians in 415 AD, the great period of Greek mathematics effectively ceased. A number of the Alexandrian students of Hypatia did manage to make their way to Athens, amongst them Proclus, and for a brief time this 'Athenian school' of mathematics held on- but it was finally closed, at the instigation of the Christians, by a decree of Justinian in 529 AD. By this time most of the original 700,000 works in the great library at Alexandria had been destroyed by the Christians - one of the greatest acts of intellectual desecration in history. The job was completed in 641 AD, with the invasion of Alexandria by Arab armies only 9 years after the death of Mohammed. It is said that the remaining volumes in the library took 6 months to burn.

In what follows we will (1) look at the historical evolution of Greek mathematics, and then (2) examine some of the key ideas and achievements that came out of it. Finally (3) I will give a brief discussion of the advances in Astronomy that were being made along with those in mathematics.

(1) BRIEF HISTORICAL NOTES

(A) Early Greek Mathematics (600-300 BC)

It was the mathematical work of the early Greek mathematicians which had such an influence on Plato- he learnt most of this from his contemporaries and friends Theatetus, Eudoxus, and Archytas (there is no evidence that Plato himself did any important original work).

We have very little of the original writings of the ancient mathematicians- just as with the philosophers, we need to rely on the commentaries of later writers, and on fragments of the originals. In Ancient times the writings of earlier mathematicians were widely available, and so historians have been able, by a process of cross-referencing, to give a sketchy picture of how things developed. This is particularly important for mathematics done before 430 BC. Some of the main sources include a commentary of Proclus, written roughly 450 AD, which was mostly occupied with discussions of Euclid, but also relied heavily on an earlier history of mathematics, now lost, written by Eudemus around 350 BC; this and other sources tell us quite a lot of what was in Eudemus's history. There are also fragments of a "General View of Mathematics" written by the Roman Geminus around 50 BC; and by various means one can infer what were the earlier sources of many of the propositions in Euclid's works (themselves written around 300 BC), which we have in essentially complete form. With all this, the following picture emerges:

(i) IONIAN SCHOOL: The life of Thales (c. 640-550 BC), the first of this school, was already recounted in the section on the pre-Socratic philosophers. In mathematics and astronomy we are aware of 2 main contributions. First, a set of geometric propositions concerning triangles and circles- the most interesting of these is the demonstration that the angle subtended by a diameter of a circle, as seen from any point on the circle not on the diameter, is a right angle (it is said that he sacrificed an ox after achieving this result). Second, the prediction of a solar eclipse which took place on May 28, 585 BC. The success of this prediction gave Thales a reputation all over the Greek world, and indeed ensured his name in history. The irony is that there is no way he could have made such a prediction with complete confidence- this would have required a knowledge and understanding of planetary dynamics far beyond the Greeks (indeed, such predictions only became possible well after Newton). What is most likely is that Thales was aware of the extensive Egyptian and Chaldean records of solar eclipses, extending over many centuries, where a pattern had been noted of the frequent repetition of solar eclipses at intervals of 18 years and 11 days (NB: frequent, but by no means always- whether a solar eclipse occurs depends on a variety of factors, and is complicated by the eccentricity of both the lunar and terrestrial orbits).

Thales taught mathematics and astronomy to a number of his pupils- it is widely believed that Anaximander (611-545 BC) was one of these, and he himself had a lively interest in astronomy (as already noted earlier, in our discussion of the pre-Socratic philosophers). Anaximander also is thought to have introduced the '*gnomon*' (also called the '*style*') to the Greek world- this was a stick or post stuck vertically into the ground, and used both as a sundial, and with considerable geometrical insight, as a means of deducing the meridian, the dates of the solstices, the inclination of the ecliptic, and finally the latitude of the gnomon itself. There is evidence that Anaximander indeed measured the latitude of Sparta in this way, and doubtless other places.

(ii) PYTHAGOREAN SCHOOL: This school was of great importance, not only because of its great impact on subsequent philosophy, but because of the real mathematical achievements of Pythagoras and his followers.

Pythagoras: Pythagoras was born around 570 BC on the island of Samos, just off the coast of Asia Minor- he may therefore have met Thales, and it is believed that at some time was taught by Anaximander. For further details of his life, see the section on pre-Socratics.

According to Pythagoras, the essential tool for the study of the cosmos was mathematics- which revealed its form. Numbers were divine, and their properties therefore revealed patterns in the cosmos- most notably the harmonic property, revealed in music, and in the 'music of the spheres'. As already noted, his remarkable philosophy, according to which the universe had to be understood in terms of basic 'forms', related to mathematical intervals, was very influential in later centuries. This was also true of his mathematics. We can group his basic results into 2 categories:

Theory of Numbers: The mathematical discoveries by Pythagoras came initially out of a study of the many properties of numbers, of their ratios, their factors, and of sequences and series of numbers. A pivotal discovery for Pythagoras was his discovery of the relation between musical harmony and simple fractional properties of musical intervals- it seems likely that he understood the relationship to frequencies of musical notes, for he argued that the motion of the planets, and the speeds of the sun, moon, and stars, corresponded to the musical ratios of the 4th (ratio of 4/3), the 5th (3/2) and the octave (2/1). These were later extended to a set of 8 intervals by Plato. The idea seems to have been that the motion of the planets produces sounds that were impossible for us to hear. It is likely that he also understood the geometric patterns revealed by crystals in a similar way.

Pythagoras also initiated the study of what are called 'polygonal numbers', in his study of triangular numbers- his school later studied the higher polygonal numbers. Triangular numbers are numbers that can be represented by a set of counters arranged in the form of a right isosceles triangle, with each row containing one more counter than the previous one, up to a row of n counters- the triangular number is the total number of counters. Thus in modern notation, a triangular number T_n can be written as the sum $T_n = \sum_1^n m \ = 1 + 2 + 3 + ... + n$, ie., $T_n = n(n+1)/2$. The polygonal numbers are defined analogously, in terms of counters arranged in the shape of some polygon. The interest of triangular numbers for Pythagoras lay in the connection to the geometric properties of triangles (such as that expressed in the famous theorem of Pythagoras- see below). In investigations like these, Pythagoras and his followers discovered quite a few geometric analogues of results for numbers, which could be expressed in the form of different sorts of sums over numbers, or their squares, etc. It is clear that the remarkable patterns that emerged were a source of great inspiration to them, both in finding new mathematical results, and in their broader philosophical ideas.

The discovery of irrational numbers by Pythagoras (ie., numbers that cannot be written as a fraction) then came as a great shock, since it proved that geometrical figures could not be understood solely in terms of integers or their ratios. For a discussion of the proof that $\sqrt{2}$ must be irrational, see the course slides. The shock here was not only a mathematical one- it was also philosophical, since as already noted, Pythagoras felt that the numerical properties of natural phenomena (revealed in shapes, musical tones, etc.) were fundamental to the underlying form of the cosmos. The effects of this discovery were therefore profound. In the first instance it apparently led to the divorce of the study of numbers from that of geometry and measurement in the real world. One can therefore argue that this discovery, and its interpretation by the Pythagorean and later philosophical schools, played a very unfortunate role in separating Greek thinking on philosophy and mathematics from their more 'scientific' investigations of the natural world. This is a theme I will return to again later. The discovery of irrational numbers also influenced later discussions of the paradoxes coming from infinite series of numbers, most notably those of Zeno the Eleatic (see 'pre-Socratics')

The beginning of the clarification of this mess had to wait until after the invention of the calculus, in the late 17th century. From work by Cantor at the end of the 19th century, we now know that the density of irrationals on the line is infinitely greater than that of the rational fractions, even though both are infinitely dense- a thorough study of all of this leads deep into set theory, to some extraordinary properties of infinitesimals, and to the properties of 'infinity' (and there infinitely many different types of infinity). All this happened in the late 19th and early 20th centuries, and will be discussed later on.

Plane and Solid Geometry: It is clear that Pythagoras and his later followers were much preoccupied with the relation of numbers to plane and solid geometry, in various ways. The music of the spheres, the relations between different regular solids, and planar figures, were for them merely a hint of deeper structures in the cosmos. They were also interested in the relationship to physical and even aesthetic properties- another discovery credited to them is the understanding of the ratio known as the 'Golden ratio'. One of the more controversial questions here is to whom can be attributed the discovery of the 5 regular solid figures discussed by Plato (these are solid geometrical figures whose vertices all lie on the surface of a sphere). There is good evidence that Pythagoras may have already found all of these, although Plato attributes the discovery that the octahedron and icosahedron were in this class to his contemporary and friend Theaetetus, and the cube, pyramid, and dodecahedron to Pythagoras.

Most of the geometric work of Pythagoras centred on the study of planar geometric figures (triangles, parallelograms, circles, etc), and many of the results in the first 2 volumes of Euclid's 11-volume work (and some in the 5th book) were first found by the Pythagoreans. The most famous of these results is of course 'Pythagoras's theorem', taught in all schools, according to which the length c of the hypotenuse of a right triangle is related to the lengths a and b of the other 2 sides, as $a^2 + b^2 = c^2$. The proof later related by Euclid is certainly not the one used by Pythagoras, and various ways that he may have proved it have been suggested by historians.

Of the later Pythagoreans, only Archytas, a contemporary of Plato, contributed anything really notable. He was best known in antiquity for having solved the then notorious 'Delian problem' (named because it is said to have originated in a prophecy from the oracle of Apollo at Delos), that of 'duplicating the cube' (ie., finding the ratio between the sides of 2 cubes whose volume differs by a factor of 2). He was a native of Tarentum, and had great political influence there and elsewhere in the Greek world- at one time he used this influence to save the life of Plato

The Pythagorean school initiated 2 enormously important developments in mathematics. The first was the style of rigorous deduction in mathematics- although such deduction had existed previously, the first attempts to make it systematic appear with Pythagoras. The 2nd great development was the attempt to link the study of numbers with that of geometry, and the systematic study of the property of numbers, to launch what we now call number theory. A modern scientist, looking at how Pythagoras influenced later ideas in physics and mathematics, can only be quietly amazed. The proof of the existence of irrational numbers is one of the greatest landmarks in the history of mathematics, embodying not only the discovery of a huge new mathematical territory, but also a very novel style of deductive argument.

(iii) ATHENIAN SCHOOL: This sub-section is unfinished, and will be inserted later

(B) 1st Alexandrian School (300-30 BC)

The 1st Alexandrian school had its heyday from roughly 300-200 BC; Euclid, Archimedes, and Apollonius were the most distinguished teachers at the famous library there, and they trained many students. The library at Alexandria was founded by Ptolemy, who had been one of Alexander's generals and closest colleagues during his famous campaigns- after Alexander's death, Ptolemy seized control of the Egyptian part of Alexander's empire. In founding the library, in 313 BC, Ptolemy founded what was essentially the world's first university, for he not only began to amass a huge collection of ancient manuscripts from all over the Alexandrian world (amounting already to 600,000 manuscripts only 40 years later), but also brought some of the finest scholars to Alexandria to work and teach there. The result was that the best mathematicians moved to Alexandria, from Athens and elsewhere. The spirit of this institution was perhaps summed up by Euclid's famous retort to a student when asked what use his theorems were... "give him 3 pennies, since he must make profit out of what he learns". Although the 1st Alexandrian school continued until about 30 BC, its greatest creative period effectively came to a close with the death of Archimedes in 212 BC.

Euclid (c. 330-270 BC): Not much is known about the life of Euclid- indeed, what we do know has been a subject of considerable debate. Euclid is most famous for his "*Elements*", which was not about the elements in the sense used by Empedocles or by modern chemists- it was instead a systematic treatment of most of the mathematics known at that time. Nowadays one can buy the entire *Elements* in heavily annotated editions from various publishers around the world- Euclid however divided it into 13 books, each dealing with different aspects of mathematics. It is certainly one of the most influential books in human history- the style of proofs given in it were central to mathematics for over 2000 years, and of overwhelming influence in the mathematics and philosophy of the Renaissance. It was still being used as a standard text in most schools and even universities in Europe until the beginning of the 19th century, and in the UK until the early 20th century, and many modern teachers of mathematics lament its passing. Its historical importance was summarized by the scientific historian van der Waerden:

"*Almost from the time of its writing and lasting almost to the present, the* 'Elements' *has exerted a continuous and major influence on human affairs. It was the primary source of geometric reasoning, theorems, and methods at least until the advent of non-Euclidean geometry in the 19th century. It is sometimes said that, next to the Bible, the* 'Elements' *may be the most translated, published, and studied of all the books produced in the Western world.*"

What is crucial to this work is the way the mathematical results were developed, using rigorous proofs based on the axiomatic method. This was the first comprehensive attempt to develop, in a logical way, the essential elements of all of mathematics as then understood, starting from a basic set of propositions. The basic idea of the axiomatic method is discussed in more detail in section 2 below. In the discussion of geometry these led to proofs of a large number of properties of simple or complex geometric shapes, both in 2 and 3 dimensions (this is one reason the work is so long). The work deals with everything in planar geometry from simple properties of trangles, parallelograms, etc., to the 'theory of proportions' (taken from the remarkable mathematician Eudoxus- this approach anticipates work even up to the 19th century in analysis); all of this, plus many applications, appear in books 1-6. In books 7-10 he goes on to number theory, dealing in book 10 with irrantioan numbers (this latter seems to have been based on work of Theatetus and others going back to Pythagoras, as well as modifications introduced by Euclid). Finally in books 11-13 he deals with solid geometry, finishing in book 13 with a discussion of the 5 'Platonic Solids', and of Eudoxus's 'method of exhaustion'. The most brilliant parts of Euclid are to be found mostly in the geometric sections of his work, where the power of the logical development is shown at its most elegant. Euclid was not the only contributor to geometry from this first Alexandrian school- indeed Apollonius (262-200 BC) made far greater original contributions a short time after him. However Euclid's *methodology* was to be crucial to the development of mathematics for over 2000 years, as well as having a central role to play in some parts of Western philosophy.

For those interested in the details of the '*Elements*', there is a set of Supplementary Notes on this work. Note that this was not the only work of Euclid that has survived - his work on Optics, for example, was of considerable importance, giving the first understanding of perspective (even the the physical parts of it are incorrect!). A couple of other more minor works have survived, including one on astronomy, but unfortunately it seems that some rather major works are lost, including a 4-volume work on conics, a treatise on music, and a work entitled '*Book of Fallacies*', which according to Proclus dealt in a colloquial way with reasoning in everyday life, by

"*enumerating in order the various kinds* [of Fallacies], *exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of the error with practical illustration.*"

We shall see the influence of Euclid over and over again as we proceed through the development of modern physics. His influence in the 17th century is shown by Newton's reluctance to develop the results in his '*Principia*' in terms of the calculus which he had used to discover them- instead everything was developed in Euclidean fashion, using geometric proofs. In the 19th century the relaxation of Euclid's famous 5th axiom led to the development of non-Euclidean geometry. In the early 20th century this led Einstein to his greatest work- the General Theory of relativity, which unified our understanding of non-Euclidean spacetime with gravity and matter.

Archimedes (287-212 BC): Archimedes was a native of Syracuse, son of the astronomer Phidias, and he spent much of his life there- his association with the 1st Alexandrian school comes because he was taught there, and was in regular touch with the school during his life. His closest associate in Alexandria was Conon of Samos, a friend with whom he regularly corresponded. Quite a lot is know about his life- he was rather well-known in the Greek world for his inventions, which were plentiful and in some cases very useful (the 'Archimedes screw' is still being used to pump water in some parts of the world!). His best-known inventions in ancient times were the lever, the compound pulley, the catapult, and the use of parabolic mirrors to focus light, which he used to set ships on fire. In Plutarch's '*Lives*', he describes 2 of these inventions. First, the use of the lever and pulley:

"*Archimedes had stated that given the force, any given weight might be moved, and even boasted, we are told, relying on the strength of demonstration, that if there were another earth, by going into it he could remove this. Hieron being struck with amazement at this, entreated him to make good this problem by actual experiment, and show some great weight moved by a small engine. He thereupon fixed upon a ship of burden out of the king's arsenal, which could not be drawn out of the dock without great labour and many men; and, loading her with many passengers and a full freight, sitting himself the while far off, with no great endeavour, but only holding the head of the pulley in his hand and drawing the cords by degrees, he drew the ship in a straight line, as smoothly and evenly as if she had been in the sea.*"

In 215 BC the Roman general Marcellus attacked Syracuse, and Archimedes was enlisted by King Heiron II to help defend the city. His use of mirrors to set the ships on fire was so effective that the Romans apparently had to abandon their ships and set about a lengthy siege, which lasted 3 years, hindered as it was by the use of catapults by the defenders. Plutarch's description of Archimedes's role in the defense includes the following passage:

"*When Archimedes began to ply his engines, he at once shot against the land forces all sorts of missiles, and immense masses of stone that came down with incredible noise and violence, against which no man could stand; for they knocked down those upon whom they fell in heaps, breaking all their ranks and files. Meanwhile huge poles were thrust out from the walls over the ships, and sunk some by great weights which they let down from on high upon them; others they lifted up into the air by an iron hand or beak like a crane's beak and, when they had drawn them up by the prow, and set them on end upon the poop, they plunged them to the bottom of the sea; or else the ships, drawn by engines within, and whirled about, were dashed against steep rocks that stood jutting out under the walls, with great destruction of the soldiers that were aboard them. A ship was frequently lifted up to a great height in the air (a dreadful thing to behold), and was rolled to and fro, and kept swinging, until the mariners were all thrown out, when at length it was dashed against the rocks, or let fall.*"

When Syracuse was finally taken, Marcellus gave orders that at all costs, Archimedes was to be spared. However a soldier apparently found him in the midst of a geometric demonstration being written in the sand- when Archimedes ignored the soldier, he was executed by him. According to the stories, Marcellus personally accorded the same treatment to the soldier in question.

The stories about Archimedes would hardly be complete without recounting how, during a visit to a public bath-house, his idea for what is now called Archimedes's theorem came to him- according to doubtless apocryphal sources, he cried "*Eurekaa*" and ran naked through the streets to his house, in order to elaborate the proof.

Archimedes left behind a considerable number of works, which include: (i) "*On plane equilibria*" (two books which deal with static mechanics- in particular, with the centres of gravity of a large number of planar objects including parabolic sections); (ii) "*Quadrature of the parabola*", then (iii) 2 books entitled "*On the sphere and cylinder*", which deal with the volumes of spheres and cylinders, and of portions sliced off them; (iv) "*On spirals*, which gives a thorough discussion of the geometry of spirals, and areas of sections enclosed by them; (v) "*On conoids and spheroids*", which exmines the volumes of sdegments of solid bodies, including paraboloids of revolution, hyperboloids of revolution, and spheroids obtained by rotating an ellipse either about its major axis or about its minor axis; (vi) The famous 2 books entitled "*On floating bodies*", which develop hydrostics, and include 'Archimedes's theorem, on the weight of a body immersed in water, and the related displacement of water; (vii) The "*Measurement of a circle*" which calculates the area of a circle by a process of inscribing ever finer triangles in it; and (vii) "*The Sandreckoner*", in which he proposes a number system capable of expressing very large numbers (up to 8×10^{16} in modern notation). He argues that this number can count the number of grains of sand which could be fitted into the volume of the universe- thereby giving his estimate of the size of the universe. Finally (ix), a most remarkable work, which in the last few decades has come to be known as the "*Palimpsest*" of Archimedes (a palimpsest is a manuscript whcih has been subsequently covered by another one). The original title is "*The Method*", and in it Archimedes describes how he gets some of his geometric results, including the role of mechanical demonstration to help him to the answer, before he finds a proof. This work was found in 1906, by the Danish philologist Johan Ludvig Heiberg, in the library of The Church of the Holy Sepulchre in Istanbul. The history of the wanderings of the Palimpsest over 2300 years, to its auction in 1998 to an anonymous bidder at Christie's in New York for $2 million, is quite extraordinary, and is recounted in a Supplementary Note.

From out point of view, the 2 most important achievements of Archimedes were (i) His laying of the foundations of what later became the calculus, by enormously extending the ideas of Eudoxus on the 'method of exhaustion'; and (ii) the first work in mathematical physics. The latter is of particular interest to us here. It will be noticed that in all the work of the pre-Socratic Greeks, as well as that of Plato and Aristotle, on the nature of the *material world*, almost no quantitative discussion appears anywhere. Even Democritus never tried to analyze, say, the volume of objects built up from different-shaped atoms; and the ideas of Plato and Aristotle never get anywhere near to quantitative discussion. From this point of view the work of Archimedes on physical problems was quite remarkable- although his contemporaries payed much attention to their applications (military and otherwise), the style and method of his investigations, and the results, were too far ahead of their time to be properly appreciated. It was not until Galileo that a similar combination of mathematics and empirical investigation was brought to bear on the physical world.

It would seem that Archimedes himself had a similar view about the importance of his purely scientific work, as compared to the inventions that made him famous. As Plutarch says:

"*Archimedes possessed so high a spirit, so profound a soul, and such treasures of scientific knowledge, that though these inventions had now obtained him the renown of more than human sagacity, he yet would not deign to leave behind him any commentary or writing on such subjects. Instead, repudiating as sordid and ignoble the whole trade of engineering, and every sort of art that lends itself to mere use and profit, he placed his whole affection and ambition in those purer speculations where there can be no reference to the vulgar needs of life- studies, the superiority of which to all others is unquestioned, and in which the only doubt can be whether the beauty and grandeur of the subjects examined, of the precision and cogency of the methods and means of proof, most deserve our admiration.*"

Thus Archimedes understood perfectly well what was important in his work, and was under no delusions about how his *oeuvre* was regarded by most of his contemporaries. As we shall see, not much has changed since then- the obsession of modern media (TV, newspapers, etc), politicians, and business, is still in the short-term, particularly military, applications of research in physics and the other sciences. In many countries today (eg., Canada, which has never had any government policy in science and technology, nor anyone in government who has known anything about science) there is almost no understanding whatsoever of the role that pure research has to play in the longer-term evolution of society. This is curious, given that the advance and even survival of many of the great civilisations in the past has depended crucially on their development of new science- and things are no different today.

Apollonius (262-200 BC): Apollonius was a pioneer in the study of 3-d solid geometry, and responsible for the the elaborate and well-nigh complete theory of conics and conic sections (ellipses, parabolae, hyperbolae). His work is of less direct importance to us, but it was famous in the Greek and Arabic worlds, and was a triumph of the Euclidean method of exhaustive geometric proof. The famous work "*Conics*" came in 8 books, of which 7 have survived (4 in the original Greek, and 3 others from Arab translations); at least 6 other works were written by Apollonius, some of which have come down to us from the Arabs.

Apollonius was born in Perga, which today is known as Murtana and is in Antalya, in Turkey. Perga was an important cultural centre at this time, and was also the place of worship of the Greek goddess Artemis (who herself certainly originated from farther East). When he was a young man, Apollonius went to Alexandria where he studied under the followers of Euclid; he later taught there himself. His character was later contrasted with that of Euclid (who was reputed to be very modest and equable); Apollonius himself was apparently a rather difficult person.

I do not discuss the geometric work of Apollonius here because although it is of great interest to histroians of mathemtics and of Greek thought, it has had little role to play in the evolution of physics. The same is not so true of the history of astronomy (a subject which is now a sub-branch of physics, but which for most of its history was in important ways independent of physics). Apollonius was an important figure in the history of Greek astronomy, and he along with others used geometrical models to explain planetary theory. Ptolemy in his book "*Syntaxis* credits Apollonius with the introduction of systems of eccentric and epicyclic motion, to explain the apparent motion of the planets across the sky. This is actually incorrect- Eudoxus certainly had done some of this beforehand- but it shows to what extent Ptolemy was influenced by Apollonius. We note here that the strength of Greek interest in geometry was decisive in the eventual Ptolemaic formulation of a theory of planetary motions and of the Solar System- this Ptolemaic system, described in more detail in section 2 below, was only much later overthrown by Copernicus.

(c) 2nd Alexandrian School (30 BC - 415 AD)

Unfortunately the changes wrought by the rise of Roman militaristic power were not conducive to disinterested inquiry, and it was not until the brief flowering of the 2nd Alexandrian school, around 300 AD, that further important steps were made. In this later flowering of research, mathematicians such as Pappus (c. 300 AD) or Diophantus (c. 320 AD) continued the development of geometry - although Diophantus was more concerned with understanding the abstract properties of numbers- what later Arab mathematicians called *algebra*. After this, the collapse of Western European civilisation meant that almost all of these ideas were lost, except in the Arab world. During the middle ages Arab mathematicians made very important advances, and also preserved many of the written works of the Greeks (although much was lost in the fire that destroyed the great library in Alexandria, with the loss of 700,000 volumes).

Without this Arab lifeline to the Renaissance, it is hard to imagine how the modern world would have turned out.

this sub-section on the 2nd Alexandrian school is unfinished- it will be done later

(2) MATHEMATICAL IDEAS & ACHIEVEMENTS of the GREEKS

(i) Number Theory: Just a brief word here. The development of numbers went to some extent in parallel with the development of astronomy, and the roots of these are buried in prehistoric times. This is hardly surprising- it is known that a number of birds and mammals are capable of counting, and it is obvious that prehistoric man could do this. What was crucial was the development of a written notation for keeping track of numbers (ie., accounting), and for manipulating them. These skills were already possessed by the Babylonians and Assyrians, but it is perhaps surprising how much depended on having an adequate notation to describe numbers and their operations. From the appropriate slide one can see how this notation developed, but the history is long and tortuous. It is worthwhile noting that the refinement of mathematical notation goes on today, and has often been associated with key advances in the subject. For more on this subject, see some of the references (in "supplementary material").

The interest of the Greeks in the properties of numbers began with the Pythagorean school, and with Democritus, who were interested in the extraordinary variety of properties numbers and collections of them can display. Amongst other things this led to an interest in irrational numbers, and how to approximate them, but gradually a very sophisticated understanding of number theory was built up. By the time of Plato, extensive results were known- important figures being Theodorus, Theatetus, and Eudoxus. however things really got going with the later Alexandrian schools. The advances eventually made by the Greeks in our understanding of number theory (still one of the most difficult and subtle branches of mathematics) were staggering. This work was driven purely by curiosity but led to discoveries of enormous importance. The most notable figures in all of this were Pythagoras, Archimedes, Euclid, and, much later, Diophantus, who invented what is now a whole field in modern mathematics (the field of 'Diophantine equations'). Some of this work is of considerable interest today, and it was essential to the revival of mathematics in Europe during the renaissance (Thus Fermat's famous "last theorem" of 1637, one of the most famous theorems in mathematics, and only very recently proved by Wiles, was found inscribed in Fermat's copy of the book *Arithmetica*, by Diophantus; this book was apparently carried everywhere by Fermat). There is no space here to describe all this work- those who are interested can go to the references.

(ii) Series, Approximations, and the beginnings of Calculus: The example given in the slides (Democritus's method of finding the area of a triangle, which he also generalized to find the volume of a cone) is useful because simple. More complicated examples of note were the problem of 'squaring the circle (ie., finding out π), and of 'doubling the cube' (ie., finding out $^3\sqrt{2}$). However it is important to stress that this sort of thinking led to 2 very important developments:

(a) The idea that successive approximation to some quantity could give a kind of "limiting operation". Even if the approximation never terminated, if successive terms ever more closely approximated the correct answer, then the approximation was useful. In the same way one could construct an infinite series which summed to a finite value (eg., the series $S = 1 + 1/2 + 1/4 + ...1/2^n + ...$, which in the limit of an infinite number of terms gives the simple answer $S \to 2$). The use of such series and approximations became particularly important to the Greeks once the existence of irrational numbers was understood, (a fraction does not need such manoeuvres for its evaluation, whereas an irrational one does). This is not to say that the Greeks accepted infinite series equably- the paradoxes of Zeno discussed in the slides show that they worried a lot about them- but they learnt how to deal with them.

(b) An important step was taken when attempts were made to evaluate the areas of *curves* (as opposed to simple straight line figures). This is a much harder problem, but we have seen in the slides how it was done for the circle, by dividing it into successively smaller triangles (leading to the approximate evaluation of π). In the hands of Archimedes, the techniques of dividing areas and volumes into increments, making these successively smaller, and then calculating results for areas, volumes, and centre of gravities, were developed into a fine art, with strict proofs for all the results. In this sense Archimedes invented the integral calculus, although in a less streamlined form than the later renaissance re-invention.

(iii) Greek Geometry & the Axiomatic Method: The axiomatic method was an attempt to develop in a logical way the essential elements of mathematics as then understood, starting from a basic set of propositions. These basic propositions are called '**axioms**'. An axiom is a single proposition which is simply assumed, without proof. All theorems of an axiomatic system are then derived from a few axioms using the rules of the system (the 'rules of inference').

The idea in the axiomatisation of geometry is that one specifies the relations between a set of primitive entities by the axioms (in the case of geometry these are points and lines), ie., by a set of rules. At the same time as fixing the rules of operation with the primitive entities, one can also provide an *interpretation* of the system, by specifying how to relate the objects/entities to objects in the real world- such as geometrical figures, in the case of geometry. However, what was crucial to the whole exercise, is that this latter step is *not necessary.* In other words, one can deal with propositions about points and lines, derived using the axiomatic method from a primitive set of propositions (the axioms), without having the slightest idea what these words and propositions might refer to.

The imaginative leap to such an abstraction was quite prodigious. In the 21st century we can think about it as follows- imagine that the axioms are provided to a computer, in the form of a set of instructions about what operations are allowed on a set of primitive identities which the computer can call 'birds' and 'bees' (but which we might instead like to call 'points' and 'lines'). The crucial (and perhaps non-intuitive) thing is that any 'meaning' of the terms "point" or "line", and all statements about them, is entirely acquired via the axioms. The 'meaning' nowadays is usually taken to refer to an *interpretation* of the basic entities and of the theorems in the logical system, by connecting them to objects either in some larger logical system, or to objects in the real world. But all we really have in this logical system is the theorems- this is just the set of all propositions in the logical system that can be derived from the axioms using the rules of inference. These propositions are 'true' if they can be derived from the axioms, and false if their contrary can be derived. Thus true means 'derivable from the axioms'.

If we then wish to interpret these as objects in the real world (eg., as lines, points, etc., in 3-d space) then we are at liberty to do so. However, in the modern view, the only way of deciding if this interpretation is a correct one (in the sense that true propositions about objects in the logical system are also true of the corresponding objects in the real world) is by experimentally checking in the real world. If this correspondence is valid, then we can if we wish talk about the meaning of the propositions in the logical system as though they really corresponded to statements about the real world.

As a measure of what an enormous step this was, we can look at what happened to one of the features discovered by Euclid that perplexed him. This is the story of the famous "5th axiom", or the "axiom of parallels". This axiom states that

"Given a straight line, and a point not on the line, there is one and only one line passing through this point which is parallel to the first line."

Another way to put this is that there is only one line through the point which will not cross the first line, if they are both extended to infinity. What bugged Euclid, and hundreds of mathematicians for 2000 years after him, is that this axiom seemed on the face of it to be unnecessary. The proposition seemed so self-evidently true of real figures that it was hard to see how it didn't follow from basic axioms defining straight lines, points, parallel, etc. And yet Euclid took a rigorous approach and found that he could not prove it from these other axioms- it was necessary to add it as an independent axiom. Thus arose the famous problem- how to find a proof that the 5th axiom did follow from the others. Many such 'proofs' were devised, but they were all incorrect.

Finally, in 1829, the shattering conclusion was published by Lobachevski, and independently in 1832 by Bolyai (although it seems Gauss had already made the discovery in 1824). These mathematicians took the bold step of *denying* the 5th axiom, and showing that one could get not one but an infinite variety of other "non-Euclidean" geometries. Finally, in the 20th century, Einstein capped it all by showing that the universe was indeed described by non-Euclidean geometry! We get to this later, in discussing relativity. But the point to be emphasized here is that Euclid, in developing the axiomatic method, and sticking to it, had opened the door to possibilities which could not have been arrived at from observations of the world, or intuition based on experience.

At the time Euclid did this work, the implications were too far-reaching for the Greek world. In fact the work of Pappus 600 years later was still further developing Euclidean geometry; and Descartes, Newton, Laplace, Maxwell, etc., never questioned it as a description of the real world.