REMARKS ON FOURIER TRANSFORMS

This is a big topic, which is discussed in countless books and web entries. Here I just want to make a few points, connected with their use in solving simple dynamical problems.

(a) FOURIER SERIES

: You are all fairly familiar with these, so I just recall some of the main points. Suppose we have a function $f(t)$ defined between times $t=0$ and $t = T_0$. Without loss of generality we can change variables so that the variable is changed to the angle θ, defined between $\theta = 0$ and $\theta = 2\pi$ (so that $2\pi t/T_0 = 2\pi$). We can define the Fourier series representation of $f(\theta)$ ~~via~~ a sum over positive integers:

$$f(\theta) = \tfrac{1}{2} A_0^f + \sum_{n=1}^{\infty} \left(A_n^f \cos n\theta + B_n^f \sin n\theta \right) \tag{1}$$

and the coefficients are

$$\left.\begin{aligned} A_0^f &= \frac{1}{\pi}\int_0^{2\pi} d\theta\, f(\theta) \\ A_n^f &= \frac{1}{\pi}\int_0^{2\pi} d\theta\, f(\theta)\cos n\theta \\ B_n^f &= \frac{1}{\pi}\int_0^{2\pi} d\theta\, f(\theta)\sin n\theta \end{aligned}\right\} \tag{2}$$

Here we notice the <u>orthogonality</u> of the set of functions $\cos n\theta$ and $\sin n\theta$; we have

$$\left.\begin{aligned} \frac{1}{\pi}\int_0^{2\pi} d\theta \cos n\theta \cos m\theta &= \frac{1}{\pi}\int_0^{2\pi} d\theta \sin n\theta \sin m\theta = \delta_{nm} \\ \frac{1}{\pi}\int d\theta \cos n\theta \sin m\theta &= 0 \qquad \forall\ n, m \end{aligned}\right\} \tag{3}$$

where δ_{nm} is the Kronecker delta-fn, ie.,

$$\left.\begin{aligned} \delta_{nm} &= 1 \quad n = m \\ &= 0 \quad n \neq m \end{aligned}\right\} \tag{4}$$

What these eqtns are telling us is that

(i) We can expand the function in terms of a set of functions, just like we can expand a vector $\underline{V}$ in terms of a set of basis vectors $\hat{\underline{x}}, \hat{\underline{y}}, \hat{\underline{z}}$.

(ii) In the same way that basis vector sets like $\hat{\underline{x}}, \hat{\underline{y}}, \hat{\underline{z}}$ are orthonormal (ie., $\hat{\underline{x}}\cdot\hat{\underline{y}} = \hat{\underline{x}}\cdot\hat{\underline{z}} = 0$, and $\hat{\underline{x}}\cdot\hat{\underline{x}} = 1$), the same is true of the functions $\{\cos n\theta, \sin n\theta)$ in eqtns (1)-(3).

The question that remains here is whether this set of "basis functions" is <u>complete</u>, ie., whether we can expand any function $f(\theta)$ in terms of them. Although I won't discuss this further, the answer is that we can provided the functions $f(\theta)$ are sufficiently smooth — a discontinuity in $f(\theta)$ will cause problems.

Most of the time it is actually more convenient to use complex Fourier series — the results in (1)-(3) assume that $f(\theta)$ is real. We then write a sum over <u>all</u> integers, in the form

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta} \qquad (5)$$

with coefficients

$$f_n = \int_0^{2\pi} \frac{d\theta}{2\pi} f(\theta) e^{-in\theta} \qquad (6)$$

where we observe that

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-im\theta} e^{in\theta} = \delta_{mn} \qquad (7)$$

so that the basis function $\{e^{in\theta}\}$ again are used to expand $f(\theta)$. If $f(\theta)$ is real we can rewrite all of this in the form of (1)-(3)

(b) <u>FOURIER TRANSFORMS</u> : These can be viewed very simply as the extension of the idea of the Fourier series to cover the entire line (and eventually the complex plane), instead of just the interval $0 \leq \theta \leq 2\pi$.

Notice first that we can rewrite the results in (5)-(7) over an interval $-T_0/2 < t < T_0/2$ (using a symmetric interval for convenience), as:

$$\left.\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} f_n e^{2\pi i n t/T_0} \\ f_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} dt\, e^{-2\pi i n t/T_0} f(t) \end{aligned}\right\} \qquad (8)$$

noting that

$$\frac{1}{T_0}\int_{T_0/2}^{T_0/2} dt\, e^{-2\pi i m t/T_0} e^{2\pi i n t/T_0} = \delta_{mn} \qquad (9)$$

Now let's extend the interval T_0 to infinity, by defining a new variable ω, which in the limit as $T_0 \to \infty$ becomes a continuous variable rather than a discrete one. We write:

$$\left.\begin{aligned} \omega &= \frac{2\pi n}{T_0} \\ f(\omega) &= T_0 f_n \end{aligned}\right\} \qquad (10)$$

Then in the limit $T_0 \to \infty$, we get sums converted to integrals, in the form

$$\left.\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega)\, e^{i\omega t} \\ f(\omega) &= \int_{-\infty}^{\infty} dt\, f(\omega)\, e^{-i\omega t} \end{aligned}\right\} \qquad (11)$$

where our orthonormality relation now becomes

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, e^{-i\omega t'} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega(t-t')} = \delta(t-t') \qquad (12)$$

where $\delta(t)$ is just the Dirac delta-function, defined by

$$\left.\begin{aligned} \delta(t-t') &= 0 \qquad (t \neq t') \\ \int_{-\infty}^{\infty} dt\, \delta(t-t') &= 1. \end{aligned}\right\} \qquad (13)$$

and indeed we have
$$\int_{-\infty}^{\infty} dt\, f(t)\, \delta(t-t') = f(t') \qquad (14)$$

for an function $f(t)$.

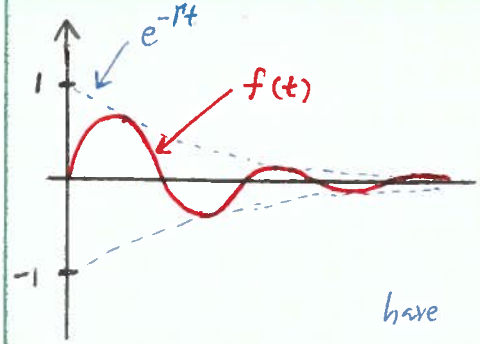
Thus, what we have here is a means of "spectrally resolving" a function into its different Fourier components. If $f(t)$ is a function of time t, then ω is a frequency, and $f(\omega)$ is a measure of the contribution of the frequency component of frequency ω, to $f(t)$, of form $f(\omega)\, e^{i\omega t}$. To resolve a signal $f(t)$ into its different Fourier or frequency components is to find $f(\omega)$.

EXAMPLE : Consider the function

$$f(t) = \theta(t)\, e^{-\Gamma t} \sin \Omega_0 t \qquad (15)$$

where the function $\theta(t)$ is the Heaviside or "step" function, defined by

$$\theta(t) = \begin{cases} 0 & (t<0) \\ 1 & (t \geq 0) \end{cases} \qquad (16)$$



so that $f(t)$ describes a damped oscillation, suddenly switched on at time $t=0$. This is the form we get for a damped oscillator suddenly kicked, with a "unit kick", at $t=0$.

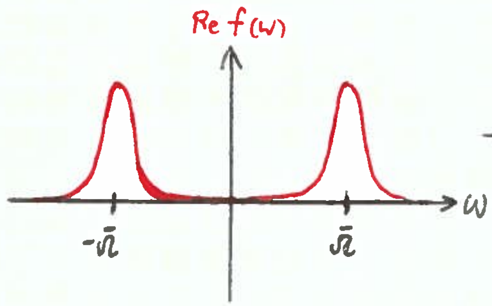
We want the Fourier transform of this. We have

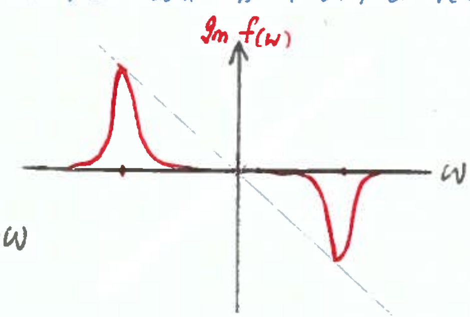
$$\left.\begin{aligned} f(\omega) &= \int_{-\infty}^{\infty} dt\, f(t)\, e^{-i\omega t} \\ &= \int_0^{\infty} dt\, e^{-i(\omega - i\Gamma)t} \sin \Omega_0 t \\ &= \frac{1}{2i} \int_0^{\infty} dt\, e^{-i(\omega - i\Gamma)t} \left(e^{i\Omega_0 t} - e^{-i\Omega_0 t} \right) \end{aligned}\right\} \qquad (17)$$

where we split $\sin \Omega_0 t$ into its positive & negative frequency components. This integral is simple to do, & we get

$$\left.\begin{aligned} f(\omega) &= \frac{1}{2}\left[\frac{1}{\omega - \Omega_0 - i\Gamma} - \frac{1}{\omega + \Omega_0 - i\Gamma} \right] \\ &= \frac{\Omega_0}{(\omega - i\Gamma)^2 - \Omega_0^2} \\ &= \frac{\Omega_0}{(\omega^2 - \Gamma^2 - \Omega_0^2) + 4\Gamma^2\omega^2} \left[(\omega^2 - \Gamma^2 + \Omega_0^2) - 2i\Gamma\omega \right] \end{aligned}\right\} \qquad (18)$$

where in the last form we write the result as a sum of real and imaginary parts.





These 2 functions, Re $f(\omega)$ and Im $f(\omega)$, are plotted at left, in the case where $\Gamma \ll \Omega_0$, i.e., of relatively small damping.

We notice that there is a frequency shift caused by the damping; the peak of the Fourier transform is at a frequency $\bar{\Omega}$ given by

$$\bar{\Omega} = \sqrt{\Omega_0^2 + \Gamma^2} \tag{19}$$

Many tables of Fourier transforms can be found.

USEFUL RESULTS for FOURIER TRANSFORMS: We have introduced the topic of Fourier transforms as a means of spectrally analyzing a function. However there are other ways to think about it. One is that it is a kind of "integral transform", of which there are many that are useful in physics & mathematics. Other examples include Laplace transforms, Mellin transforms, Hilbert transforms, Hankel transforms, Abel transforms, and Legendre transforms.

The key reason that scientists use integral transforms is that when applied to some classes of differential or integral (or integro-differential) eqtn., they transform it to a much simpler form, where it can be solved — in essence, the function or eqtn is mapped to a different domain in which it has simpler behaviour. Once it is solved, the solution is then transformed back, via the inverse transform, to the original domain.

In the case of Fourier transforms, we can see why they make differential & integro-differential eqtns so easy to solve by considering the following results (which are easy to prove):

(i) The Fourier transform $\mathcal{F}[f'(t)] \equiv \mathcal{F}[df/dt] \equiv \int_{-\infty}^{\infty} dt \, \frac{df}{dt} e^{-i\omega t}$

is given by: $$\mathcal{F}[f'(t)] = i\omega \mathcal{F}[f(t)] = i\omega f(\omega) \tag{20}$$

and likewise: $$\mathcal{F}[d^n f(t)/dt^n] = (i\omega)^n f(\omega) \tag{21}$$

(ii) Integrals are a little messier, We have

$$\mathcal{F}\left[\int dt' f(t')\right] \equiv \int_{-\infty}^{\infty} dt \, e^{-i\omega t} \int_{-\infty}^{t} dt' \, f(t') \tag{22}$$

and this turns out to be

$$\mathcal{F}\left[\int^{t} dt' f(t')\right] = \left[\frac{1}{i\omega} + \pi \delta(\omega)\right] f(\omega) \tag{23}$$

These are the 2 most important results — you will find it helpful to derive them. There are a number of others which follow fairly straightforwardly from these — the most important involve Fourier transforms of $f(t)$ multiplied by a polynomial in t, by an exponential, and also Fourier transforms of products of different functions, or of convolutions of them.

Thus, for example, we have

$$\left.\begin{aligned} \mathcal{F}[e^{i\Omega_0 t} e^{-\Gamma t} f(t)] &= \int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega t}\, e^{i(\Omega_0 + i\Gamma)t} \\ &= f(\omega - (\Omega_0 + i\Gamma)) \end{aligned}\right\} \qquad (24)$$

and also, by the inverse reasoning, that

$$\mathcal{F}[f(t-\tau_0)] = f(\omega)\, e^{-i\omega\tau_0} \qquad (25)$$

sometimes called the "shift identity".

If we consider products or convolutions of functions, then we can easily get the following results:

(iii) Convolutions like $\quad f(t) = \int_{-\infty}^{\infty} dt'\, g_1(t-t')\, g_2(t') \qquad (26)$

have Fourier transform

$$f(\omega) = g_1(\omega)\, g_2(\omega) \qquad (27)$$

(iv) Parseval's theorem on integrals over products. This is usually written as

$$\int_{-\infty}^{\infty} dt\, g_1(t)\, g_2(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, g_1(\omega)\, g_2(\omega) \qquad (28)$$

and is easily proved by rewriting each of $g_1(t)$ and $g_2(t)$ in their Fourier expanded form.

Many analogous results can be derived for other integral transforms. Thus, e.g., in the various results for Laplace transforms, defined by

$$f(s) = \int_0^{\infty} dt\, f(t)\, e^{-st} \qquad (29)$$

we find relationships similar to those in eqtns. (20) - (28), although they differ in detail.