BRIEF NOTE ON PERTURBATION METHODS

The following is a VERY LIMITED introduction to a very large and intricate subject - it should do no more than make you feel comfortable with the general idea.

The basic idea here is to try to deal with finding the solution to a differential eqtn., or an integral eqtn., or an integral, or some other eqtn for which we wish to find the solution, by starting from another "nearby" eqtn for which we do have the solution, and then expressing the solution to the original eqtn in terms of the difference between the old and new eqtns. Roughly speaking, we can imagine we are faced with a problem like

$$\left.\begin{aligned} \hat{F}[x(t)] &= 0 \\ \hat{F}_0[x_0(t)] &= 0 \end{aligned}\right\} \qquad (1)$$

and

$$\hat{F} - \hat{F}_0 \sim O(\epsilon) \qquad (\epsilon \ll 1.) \qquad (2)$$

where the two operators $\hat{F}$ and $\hat{F}_0$ differ by only a small amount. We then write the solution $x(t)$ in the form

$$x(t) = x_0(t) + \sum_{n=1}^{\infty} \epsilon^n x_n(t) \qquad (\epsilon \ll 1) \qquad (3).$$

The idea is to then solve the problem systematically by equating terms of the same order in ϵ; since ϵ is a variable, this must be correct. The method is best illustrated using examples.

<u>EXAMPLE 1</u> : Consider the simple algebraic eqtn:

$$x^3 - 4.001\,x + 0.002 = 0 \qquad (4)$$

This eqtn, a cubic eqtn., can be written in the form

$$x^3 - (4+\epsilon)x + 2\epsilon = 0 \qquad (5)$$

with $\epsilon = 10^{-3}$. But let's now solve eqtn (5) with ϵ now assumed to be a <u>variable</u> (always assuming $\epsilon \ll 1$). We substitute

$$x = x_0 + \sum_{n=1}^{\infty} \epsilon^n x_n \qquad (6)$$

into (5), we get the result:

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots)^3 - (4+\epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots) + 2\epsilon = 0 \qquad (7)$$

ie.,

$$(x_0^3 - 4x_0) + (3x_0^2 x_1 - 4x_1 - x_0)\,\epsilon + \left(3(x_0^2 x_2 + x_1^2 x_0) - 4x_2 - x_1\right)\epsilon^2 + \dots + 2\epsilon = 0 \qquad (8)$$

where in (8) we do the expansion of each term in powers of ϵ. Now, since ϵ is a variable, for (7) or (8) to be satisfied we must have terms to any order in ϵ also equal to zero. Thus, grouping these powers, we have

$$\left.\begin{aligned} \epsilon^0 &: \quad x_0^3 - 4x_0 = 0 \\ \epsilon^1 &: \quad 3x_0^2x_1 - (4x_1 + x_0) + 2 = 0 \\ \epsilon^2 &: \quad 3(x_0^2x_2 + x_1^2x_0) - (4x_2 + x_1) = 0 \end{aligned}\right\} \tag{9}$$

and so on. Now the point is that when $\epsilon \to 0$, we are left with x_0, which is easily found - this is the UNPERTURBED EQTN., and we have

$$x_0^3 - 4x_0 = 0 \quad \Rightarrow \quad x_0 = 0, \pm 2. \tag{10}$$

We can now find perturbation expansions for x around these unperturbed solutions in (10). For the 3 different roots $x_0 = 0, \pm 2$, we find the following.

$$\left.\begin{aligned} &(i)\ x_0 = 0 : && x_1 = \tfrac{1}{2},\ x_2 = -\tfrac{1}{8},\ \text{etc.} && x(t) = \tfrac{1}{2}\epsilon - \tfrac{1}{8}\epsilon^2 + \cdots \\ &(ii)\ x_0 = 2 : && x_1 = 0,\ x_2 = 0,\ \text{etc.} \Rightarrow && x(t) = 2 \\ &(iii)\ x_1 = -2 : && x_1 = -\tfrac{1}{2},\ x_2 = \tfrac{1}{8},\ \text{etc} \Rightarrow && x(t) = -2 - \tfrac{1}{2}\epsilon + \tfrac{1}{8}\epsilon^2 + \cdots \end{aligned}\right\} \tag{11}$$

or, in other words, the root at $x_0 = 2$ is completely unshifted by the small perturbation, and the other 2 roots are shifted in opposite directions. Finally, we let $\epsilon = 10^{-3}$.

<u>EXAMPLE 2</u> : We pick the example of the perturbing effect of general relativistic effects on Newtonian dynamics of a particle orbiting in a central potential. The equation is now a differential eqtn., which is written as $L[u] = 0$, where

$$\left.\begin{aligned} \hat{L}[u] &= (u'' + u - V_0/\ell^2) + \epsilon u^2 \\ &\equiv L_0[u] + \epsilon u^2 \end{aligned}\right\} \tag{12}$$

and where in the G.R. problem, $u(\phi) = 1/r(\phi)$, and $\epsilon = -3V_0$. One makes the expansion

$$u(\phi) = u_0(\phi) + \sum_{k=1}^{\infty} u_k(\phi)\,\epsilon^k \tag{13}$$

and substitutes this into (12); from a comparison between (12) and (13, we see that

$$\hat{L}_0 \, u_0(\phi) = 0 \tag{14}$$

and this equation is solvable ; one has the solution

$$u_0(\phi) = V_0/_{\ell^2} \, (1 + e \cos \phi) \tag{15}$$

where e is a constant. The expansion of (12), using (13), gives

$$\hat{L} \, u_0 + \epsilon(\hat{L}_1 u_0 + \hat{L}_0 u_1) + \epsilon^2(\hat{L}_1 u_1 + \hat{L}_0 u_2) + \cdots = 0 \tag{16}$$

and each coefficient in the powers of ϵ to zero, we get

$$\left.\begin{aligned} \hat{L}_1 u_0 + \hat{L}_0 u_1 &= 0 \\ \hat{L}_1 u_1 + \hat{L}_0 u_2 &= 0 \\ \text{etc} \end{aligned}\right\} \tag{17}$$

or, written out explicitly :

$$\left.\begin{aligned} u_0^2 + (u_1'' + u_1 - V_0/\ell^2) &= 0 \\ u_1^2 + (u_2'' + u_2 - V_0/\ell^2) &= 0 \\ \text{etc} \end{aligned}\right\} \tag{18}$$

To see how this works out in practise, you can look at the notes on the Schwarzchild problem.

We see that the general procedure is fairly straightforward. Given some equation in the form (1), we write

$$\hat{F} = \hat{F}_0 + \epsilon \hat{F}_1 \tag{19}$$

and let the solution of eqtn (1) be written as a perturbation expansion in the small parameter ϵ, in the form of (3), ie

$$\left.\begin{aligned} x(t) &= \sum_{n=0}^{\infty} \epsilon^n x_n(t) \\ \text{with} \quad \hat{F}_0 \, x_0(t) &= 0. \end{aligned}\right\} \tag{20}$$

In the case where we know $x_0(t)$, we can then set up a series of coupled equations to solve for the $x_n(t)$. Finally, substitute in the appropriate value of ϵ, to get the final result.