ORBITING A SCHWARZCHILD BLACK HOLE

Although you are obviously not expected to know about General Relativity in this class (or even special relativity), it is possible to give a treatment of orbital motion in General Relativity that is elementary enough that one can think about it in the same language as we have been using for central field orbital motion in Newtonian mechanics.

In what follows I will give a treatment of orbital motion in General Relativity (henceforth "GR") around a non-rotating central mass.* If this mass is sufficiently compact it will become a "Schwarzchild black hole" (named after K. Schwarzchild, who found the solution while a soldier on the Russian front in 1916).

THE NEWTONIAN PROBLEM: We recall that in Newtonian mechanics, the radial equation in a central field is

$$m\ddot{r} + \frac{\partial U_R}{\partial r} = 0 \qquad (1)$$

with angular eqtn:

$$mr^2\dot{\phi} = \ell = \text{const} \qquad (2)$$

and where the effective potential

$$U_R(r) = V(r) + \ell^2/2mr^2 \qquad (3)$$

In what follows we will assume that

$$\left.\begin{aligned} V(r) &= -V_0/r \\ m &= 1 \end{aligned}\right\} \qquad (4)$$

for the Newtonian problem. We know of course that $V_0 = GMm$, where M is the central mass, m is the orbiting mass, and $M \gg m$ (if we want, we can write things in terms of a reduced mass). I will make $m = 1$ to simplify things, (since it is just a constant multiplying all the equations); if you like you can imagine we are dealing with a "test mass", with $m = 1$ in some set of units.

From the conservation of energy in Newtonian mechanics we then have

$$\dot{r}^2 + 2U_R(r) = 2E \qquad (5)$$

and we wish to compare this with the analogous eqtn. in GR.

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* Motion around a rotating mass in GR is described by the very complicated "Kerr solution", far more difficult a problem; spacetime is "twisted" by the angular momentum of the central mass, and radial & angular motions are coupled.

SCHWARZCHILD RADIAL EQTN. of MOTION : The analogue in GR to the Newtonian eqtns. in (2), (5) can be written in the form

$$r^2 \dot{\phi} = \ell = \text{const.} \tag{6}$$

$$\dot{r}^2 + \mathcal{V}_R^2(r) = \mathcal{E}^2 = \text{const.} \tag{7}$$

where here

$$\mathcal{V}_R^2(r) = \left(1 - 2V_0/r\right)\left(1 + \ell^2/r^2\right) \tag{8}$$

(recall again we are assuming $m = 1$). This looks a little different from the Newtonian eqtn., and below I will explain how it reduces to the Newtonian result in the appropriate limit. For those of you who want to compare what I have written with books, note that I have defined things as follows:

$$V_0 = GM/c^2 \tag{9}$$

$$\mathcal{E} = E/mc^2 \equiv E/c^2 \tag{10}$$

where c is the velocity of light; and then finally, to simplify the eqtns., I have used units where

$$c = 1. \tag{11}$$

We see that $\mathcal{E}$ is actually the energy E measured in relativistic units (where a unit of energy is mc^2).

In any case, we can rewrite the energy equation in terms of $u(\phi) = 1/r(\phi)$ in exactly the same way as is done in Newtonian mechanics. We first note that

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{\ell}{r^2}\frac{dr}{d\phi} \tag{12}$$

from (6), so that we have

$$\left(\frac{\ell}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{\ell^2}{r^2} + 1 = \mathcal{E}^2 + 2V_0\left(\frac{1}{r} + \ell^2/r^3\right) \tag{13}$$

or, rewriting in terms of $u = 1/r$ in the usual way, that

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = 1/\ell^2\left(\mathcal{E}^2 - 1\right) + 2V_0\left(\frac{u}{\ell^2} + u^3\right) \tag{14}$$

with $u = u(\phi)$. We get rid of the constants in this eqtn by differentiating with respect to ϕ, to get

$$\frac{d^2u}{d\phi^2} + u - \frac{V_0}{\ell^2} - 3V_0 u^2 = 0 \qquad (GR) \qquad (15)$$

which we can compare with the Newtonian eqtn, viz.,

$$\frac{d^2u}{d\phi^2} + u - \frac{V_0}{\ell^2} = 0 \qquad (Newton) \qquad (16)$$

so that we see that the difference here lies in the term $3V_0u^2$ in the GR eqtn.

For those who want to go more deeply into this, I will give a sketch of the derivation of eqtns (6)-(8) at the end of this note; this sketch is purely for your interest, and is not part of the course material.

DYNAMICS in the SCHWARZCHILD FIELD: We now consider how we might solve (15). Actually an analytic solution can be given in terms of elliptic functions; but here we will focus on the PERTURBATIVE solution of the eqtn. (15). This is actually fairly simple — but it assumes that the perturbing term $3V_0u^2$ in (15) is small compared to the term $(u - V_0/\ell^2)$. Basically this means that we want u to be small and ℓ to be large — I will discuss the precise conditions below. Both of these conditions imply that r is large compared to some characteristic length scale, which we will derive below.

Now we notice that if the perturbing term $3V_0u^2$ is neglected, we get back the Newtonian eqtn (16), for which the solution is, as we already know:

$$u(\phi) \rightarrow u_0(\phi) = \frac{V_0}{\ell^2}(1 + e\cos\phi) \qquad (17)$$

where I let $\phi_0 = 0$ by choosing axes accordingly. Now if the perturbation is small, we can expect the change in $u(\phi)$ from $u_0(\phi)$ to be small; so let's write

$$u(\phi) \sim u_0(\phi) + \delta u(\phi) \qquad (18)$$

and substitute this into the eqtn of motion (15). This gives us

$$\frac{d^2}{d\phi^2}(\delta u) + \delta u = \frac{3V_0^3}{\ell^4}(1 + 2e\cos\phi + e^2\cos^2\phi) \qquad (19)$$

with $3V_0^3/\ell^4 \ll 1$ by assumption.

Now one can solve this differential eqtn for $\delta u(\phi)$ in various ways. We know that the general solution of the homogeneous eqtn is just $\delta u(\phi) = u_0(\phi)$, since the left-hand side of (19) is just the homogeneous operator used in finding Newton's solution $u_0(\phi)$ in (17). So one way is just to find a particular integral; and by examination of (19), one such particular integral is found to be

$$\delta u(\phi) = \frac{3V_0^3}{\ell^4}\left[1 + e\phi\sin\phi - \frac{e^2}{6}(\cos 2\phi - 3)\right] \tag{20}$$

However we can also employ a physical argument to simplify (19). We note that the 3 terms on the RHS of (19) have quite different effects as driving terms. They are all very small; but the 2nd term is in resonance with the solution $u_0(\phi)$ in (17), since it drives it at the same "frequency" as that in $u_0(\phi)$ (they both go like $\cos\phi$). This suggests that we drop the 1st & 3rd terms on the RHS of (19), to give a differential eqtn.

$$\left(\frac{d^2}{d\phi^2} + 1\right)\delta u(\phi) = \frac{6eV_0^3}{\ell^4}\cos\phi \tag{21}$$

which is a standard eqtn. for an oscillator, driven by a periodic "force" (where now the time coordinate is written as ϕ). A particular integral of (21) is

$$\delta u(\phi) = \frac{3V_0^3}{\ell^4}e\,\phi\cos\phi \tag{22}$$

Now (20) and (22) are basically saying the same thing: all terms in (20) are small, but as ϕ increases, the term $\propto \phi\sin\phi$ grows without limit – this is the resonance phenomenon.

Adding (22) to $u_0(\phi)$, we then get a solution

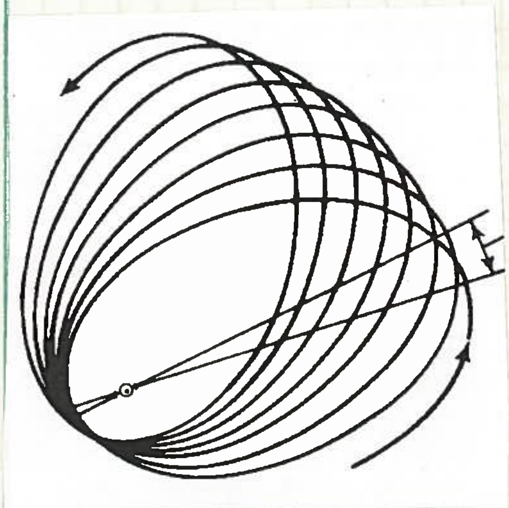
$$\left.\begin{aligned} u(\phi) &= \frac{V_0}{\ell^2}\left[(1 + e\cos\phi) + Ae\,\phi\cos\phi\right] \\ \text{where} \quad A &= 3V_0^2/\ell^2 \ll 1 \end{aligned}\right\} \tag{23}$$

Now since A is so small, we can simply write $(\cos\phi + A\phi\sin\phi) \sim \cos((1-A)\phi)$, and then the solution becomes

$$u(\phi) \sim \frac{V_0}{\ell^2}\left[1 + e\cos\left(\left(1 - 3V_0^2/\ell^2\right)\phi\right)\right] \tag{24}$$

which was Einstein's famous solution to the problem of the precessing orbit of Mercury, and the first striking confirmation of the theory of General Relativity.

To see why this solution is equivalent to a precession of the orbit of a planet around a central mass like the sun, notice that the effect of the factor of $(1-A) = (1-3V_o^2/\ell^2)$ in the argument of $\cos[(1-A)\phi]$ is to change the period of the cosine from 2π to $2\pi/[1-A]$, so that the orbit repeats it's maximum value $r(\phi)$ (or minimum value of $u(\phi)$ on an interval of $2\pi/[1-A]$ rather than 2π, ie., it no longer makes a closed orbit. This is equivalent to a precession of the orbit, with its axis slowly changing; the change in axis for each orbit is then



$$\left.\begin{aligned}\Delta\phi &= \left(2\pi - \frac{2\pi}{1-A}\right)\\ &\sim 2\pi A \quad = \quad 6\pi\, V_o^2/\ell^2\end{aligned}\right\} \qquad (25)$$

or, if we put back in all the units of c, write $V_o = GM$, and write the angular momentum ℓ in terms of the elliptical orbit parameters a (semi-major axis) and e, we then get

$$\Delta\phi = 6\pi\left(\frac{GM}{\ell c}\right)^2 = 6\pi\,\frac{GM}{c^2}\,\frac{1}{a(1-e^2)} \qquad (26)$$

The predicted precession for Mercury is very small, amounting to 42.98 arc seconds per century. Observations find a discrepancy between Newtonian theory and the observed precession of (43.11 ± 0.45) arc seconds per century. Note that the precession is far larger for pairs of orbiting neutron stars, and the theory works very well here as well.

MORE ON THIS PERTURBATIVE APPROACH : The approach we used above is a little crude, and only works if $V_o^2/\ell^2 \ll 1$. It is worth asking how one can do things better, if V_o^2/ℓ^2 is not quite so small. Then we need to use a more formal perturbative approach; we write

$$\left.\begin{aligned}u(\phi) &= u_o(\phi) + \sum_{k=1}^{\infty} u_k(\phi)\,\epsilon^k\\ &= u_o + \epsilon u_1 + \epsilon^2 u_2 + \cdots\end{aligned}\right\} \quad \epsilon \ll 1 \qquad (27)$$

and then substitute this into the differential eqtn in (15), written as

$$\hat{L}_o(u) + \epsilon \hat{L}_1(u) = 0 \qquad (28)$$

where

$$\left.\begin{aligned}\hat{L}_o(u) &= (d^2/d\phi^2 + 1)\,u(\phi) - V_o/\ell^2\\ \hat{L}_1(u) &= -3V_o\, u^2(\phi)\end{aligned}\right\} \qquad (29)$$

and where formally $\epsilon = 1$. However we then let ϵ be a free parameter, with the assumption that $\epsilon \ll 1$ to begin with, and use successive approximation. Thus, we begin by substituting $u(\phi) \sim u_0(\phi)$ into (15), noting of course that

$$\hat{L}_0(u)\, u_0(\phi) = 0. \tag{30}$$

Now substituting (27) into (28), we get

$$\underbrace{\hat{L}_0 u_0}_{=0} + \epsilon(\hat{L}_1 u_0 + \hat{L}_0 u_1) + \epsilon^2(\hat{L}_1 u_1 + \hat{L}_0 u_2) + \cdots = 0 \tag{31}$$

where from (30), the first term in (31) is zero. We now group everything according to powers of ϵ; since ϵ is an arbitrary parameter, the terms to each order in ϵ must individually be zero as well. Thus we have

$$\left.\begin{array}{ll} \hat{L}_1 u_0 + \hat{L}_0 u_1 & = 0 \\ \hat{L}_1 u_1 + L_0 u_2 & = 0 \\ \quad\text{etc} & \end{array}\right\} \tag{32}$$

The first eqtn., viz., that $\hat{L}_1 u_0 + \hat{L}_0 u_1 = 0$, when written out using $u_0(\phi)$ as given in (17), just gives (19). It is then an interesting exercise to continue on deriving higher approximations to $u(\phi)$ from this hierarchy.

Note that we would also have derived the same result for the correcting eqtn. (19) by the very naive procedure of substituting the Newtonian solution (17) into the perturbation term $3V_0 u^2(\phi)$ in (15), so that (15) reads

$$\frac{d^2u}{d\phi^2} + u(\phi) - V_0/\ell^2 - 3V_0\, u_0^2(\phi) = 0 \tag{33}$$

with the justification that $u(\phi) - u_0(\phi)$ is small, so that corrections to (33) will be very small. This procedure is justified by the perturbative method discussed here, since eqtn (33) can also be written as

$$\hat{L}_0(u_0 + \epsilon u_1) + \epsilon \hat{L}_1 u_0 = 0 \tag{34}$$

which is just the first eqtn. in (32) (recalling again that $\hat{L}_0 u_0 = 0$).

GENERAL RELATIVISTIC SETTING for THESE RESULTS

: Some of you may be interested to know where all of this comes from, ie., where do we get our original eqtns (6)-(8) from? Here I will briefly sketch this for you.

In General Relativity one starts from an assumption that the spacetime metric, which is defined by the infinitesimal interval ds at some spacetime coordinate x (where now $x = t, x_1, x_2, x_3$) is a 4 dimensional spacetime coordinate), is no longer "flat" in general. By this we mean that in the equation

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \equiv \sum_\mu \sum_\nu g_{\mu\nu}\, dx^\mu dx^\nu \qquad (35)$$

where we have used Einstein's "summation convention", that repeated indices are summed over (and where the indices μ, ν run from 0 to 3, covering the time and 3 space indices), the metric $g_{\mu\nu}(x)$ can be arbitrary. Instead of taking the flat spacetime form

$$g_{\mu\nu}(x) \xrightarrow[\text{spacetime}]{\text{flat}} \eta_{\mu\nu}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (36)$$

so that the infinitesimal interval is

$$\left.\begin{aligned} ds^2 &= \eta_{\mu\nu}\, dx^\mu dx^\nu \\ &= c^2 dt^2 - d\underline{r}^2 \end{aligned}\right\} \qquad (37)$$

where $\underline{r}^2 = dx_1^2 + dx_2^2 + dx_3^3$, we have a general $g_{\mu\nu}(x)$.

The Schwarzchild geometry, for a point mass situated at the origin, has the metric form, in spherical coordinates (r, θ, ϕ):

$$ds^2 = \left(1 - {}^{2V_0}\!/_r\right) c^2 dt^2 - \left(\frac{1}{1 - 2V_0/r}\right) dr^2 - r^2\left(d\theta^2 + \sin^2 d\phi^2\right) \qquad (38)$$

which, if we compare with the interval in flat spacetime written in spherical coordinates, having form

$$ds^2 \xrightarrow[\text{spacetime}]{\text{flat}} \eta_{\mu\nu}\, dx^\mu dx^\nu = c^2 dt^2 - \left[dr^2 + r^2\left(d\theta^2 + \sin^2 d\phi^2\right)\right] \qquad (39)$$

shows that both the time and radial spatial coordinates are distorted; we now have $g_{x_0 x_0}(x) = (1 - 2V_0/r)$ and $g_{x_1 x_1}(x) = (1 - 2V_0/r)^{-1}$, a kind of "stretching" of spacetime.*

Now from the metric in (39) one can actually calculate the eqtn of motion of a particle moving in this metric. There are various ways of doing this – here I shall use a method that is a generalization of that used in

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* Two points should be emphasized here. First, the singular (divergent) behaviour of the metric when $2V/r = 1$ is not a singularity in spacetime, but only in this choice of the metric, which is adapted to an external at spatial infinity. Second, one should note that r and t in eqtn. (39) are not the coordinate & time as measured by an observer at ∞; they have already undergone a coordinate transformation.

non-relativistic mechanics (ie, Newtonian mechanics). We write the Lagrangian for a "test mass" (of unit mass) as

$$\mathcal{L}(\dot{x}^\mu) = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \qquad (40)$$

(this can only be derived relativistically), where $\quad \dot{x}^\mu \equiv \dfrac{dx^\mu}{d\tau} \quad (41)$

and where here we take τ to be the "proper time" along a particle trajectory. If we then substitute into Lagrange's eqtns., viz.,

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \qquad (42)$$

using the metric for the Schwarzschild geometry in (38), so that from (40),

$$\mathcal{L} = c^2\left(1-\frac{2V_0}{r}\right)\dot{t}^2 - \left(\frac{1}{1-2V_0/r}\right)\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) \qquad (43)$$

then we get the eqtns of motion, as follows:

$$\left.\begin{aligned} \left(1-\frac{2V_0}{r}\right)\dot{t} &= \mathcal{E} \\ \left(1-\frac{2V_0}{r}\right)c^2\dot{t}^2 - \left(\frac{1}{1-2V_0/r}\right)\dot{r}^2 - r^2\dot{\phi}^2 &= c^2 \\ r^2\dot{\phi} &= \ell \end{aligned}\right\} \qquad (44)$$

In getting to eqtns (43), I have dropped the eqtn in $\ddot{\theta}, \dot{\theta}$, etc., because it simply demonstrates that orbital motion will be in a plane, as in Newtonian motion (so we set $\theta = \pi/2$); and I have integrated the eqtn. on $\ddot{r}, \dot{r}$, etc, w.r.t. τ, to make it simpler. The constants $\mathcal{E}$ and ℓ are simply constants — later they acquire their meaning used in (6) and (7) above later.

If we substitute the 1st and 3rd eqtns in (44) into the 2nd one, we immediately obtain eqtn. (7), used to obtain the results derived above.

There is another useful way to think about all of this, which connects up to the usual Newtonian way of analyzing the radial energy eqtn., and which also helps clarify what is going on here.

We began in eq. (6) with what looks like an energy eqtn., except that it involves the energy squared — this at first glance looks rather peculiar. To clarify this, and what is meant by energy here, we are going to analyze eq. (6) in 2 different ways.

The two ways are best characterized by writing eq. (6) in 2 different ways, as follows:

We write either

$$\dot{r}^2 + \mathcal{U}_R^2(r) = \mathcal{E}^2 \qquad (45)$$

where $\mathcal{U}_R^2(r) = (1 - 2V_0/r)(1 + \ell^2/r^2)$

or we write

$$\tfrac{1}{2}\dot{r}^2 + V_{eff}(r) = \tilde{E} = \frac{c^2}{2}(\mathcal{E}^2 - 1) \qquad (46)$$

where $V_{eff}(r) = \frac{\ell^2}{2r^2}(1 - 2V_0/r) - V_0c^2/r$

and we have re-inserted factors of c (ie, $c \neq 1$ anymore); recall that $\mathcal{E}$ was written in units of $mc^2 \equiv c^2$.

Now, as we shall see, the latter form in (46) is useful when we go to the Newtonian limit, where we let $c \to \infty$. In this case we note that

$$\mathcal{E}^2 = \frac{m^2c^2 + p^2c^2}{m^2c^4} = 1 + v^2/c^2 \qquad (47)$$

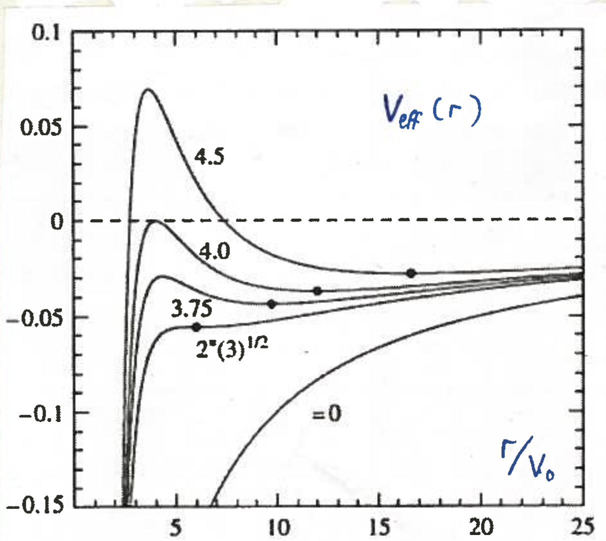
so that the right-hand side of (46) becomes $\tilde{E} = v^2/2$ (48)

and because $V_0 = GM/c^2 \xrightarrow[c\to\infty]{} 0$, we end up with

$$\tfrac{1}{2}\dot{r}^2 + V_{eff}(r) = \tilde{E}$$
$$V_{eff}(r) = \frac{\ell^2}{2r^2} - \frac{GM}{r} \qquad (49)$$

which is just the Newtonian eqtn.

However the radial eqtn. now has an extra attractive term, whether we use $V_{eff}(r)$ or $\mathcal{U}_R(r)$, we will get a quite different radial dynamics from what we found in the Newtonian case. This is because the attractive term $\sim 1/r^3$ will always overpower the centrifugal term at short range.



In the figure at left we see the behaviour of $V_{eff}(r)$ in (46), plotted as a function of the parameter

$$\tilde{\ell} = \ell / V_0 c \qquad (50)$$

To look for ~~extrema~~ extrema of this potential, we calculate dV_{eff}/dr and look for its zeroes. We have

$$\frac{dV_{eff}(r)}{dr} = \frac{V_0 c^2}{r^2} - \frac{\ell^2}{r^3} + \frac{3V_0\ell^2}{r^4} \qquad (51)$$

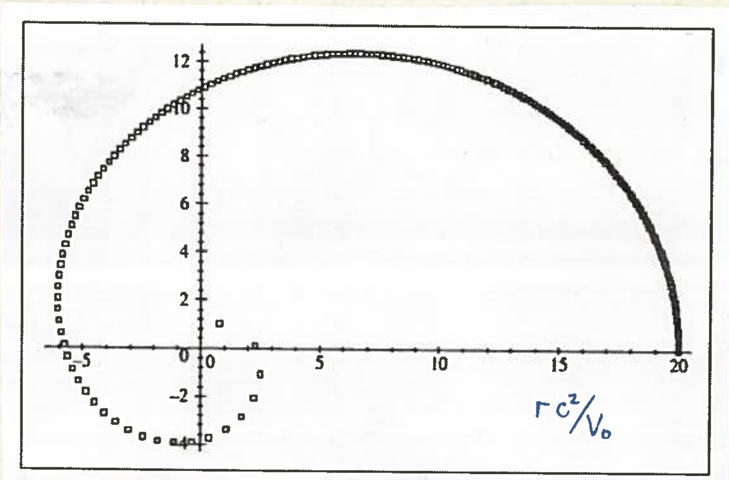
and, multiplying this eqtn by r^4, we get a quadratic eqtn for r with roots

$$r_\pm = \frac{\ell}{2V_0c^2}\left[\ell \pm (\ell^2 - 12V_0^2c^2)^{1/2}\right] \qquad (52)$$

These roots merge into one root when $\bar{\ell} = \ell/cV_0 \longrightarrow 2\sqrt{3}$ (53)

The general behavior is shown in the figure on the last page. A key result here is that when $\bar{\ell} < 2\sqrt{3}$, the test mass must fall into the central potential. Moreover, any circular orbit whose radius lies inside the potential hill in the figure must spiral into the centre – it is unstable – this happens when the radius is less than a critical value, viz., when

$$\left.\begin{aligned} r < r_{min} &= 6V_0/c^2 \\ &\equiv 6GM/c^2 \end{aligned}\right\} \quad (54)$$



rc^2/V_0

The figure at left shows a particle projected with initial angular momentum $\ell = 3V_0/c$, at a distance $r > 20\,V_0/c^2$. For the orbit to be circular, we would need a larger ℓ, given by $\ell = (20/\sqrt{17})\,V_0/c$. As a result, the particle spirals into the centre in a finite time.

So far I've said nothing here about black holes. If we analyze the trajectories of photons we would find that photons would be unstable in any conceivable orbit if

$$r < 2V_0/c^2 = 2GM/c^2 = R_s \qquad (55)$$

which is the famous Schwarzschild radius, giving the radius of the black hole event horizon.