ROTATIONAL MOTION OF RIGID BODIES

One of the most common & important problems that engineers & physicists have to deal with is the motion of rigid bodies. The translational motion is straightforward - what is less obvious is how to deal with the rotational motion, and how it couples to the translational motion. Since a lot of energy can be stored in rotational motion, the effects of releasing it can be impressive.

In what follows we first derive the key properties of rotational motion in the inertial frame - to simplify things I will assume that the centre of mass of the body is not accelerating. We then go to the frame of the rotating body itself, which of course is rotating - so the dynamics looks different in this frame.

(a) DYNAMICS IN AN INERTIAL FRAME

: We are going to need to model a rigid body in some way. This will be done here in one or other of 2 ways. We can first model it as a set of particles of mass m_j, at positions $\underline{r}_j(t)$. We can also model it as a continuous density distribution $\rho(\underline{r})$. The relationship between the two is just

$$\rho(\underline{R}) = \sum_j m_j \, \delta(\underline{R} - \underline{R}_j(t)) \qquad (1)$$

which just says that each particle is described as a point mass. Recall that the δ-function, by definition, satisfies:

$$\int d^3R \, \delta(\underline{R} - \underline{R}_j) = \underline{R}_j \qquad (2)$$

so that we see that the following relations hold:

- total mass: $$\left. \begin{aligned} M = \int d^3R \, \rho(\underline{R}) &\equiv \int d^3R \sum_j m_j \, \delta(\underline{R} - \underline{R}_j) \\ &= \sum_j m_j . \end{aligned} \right\} \qquad (3)$$

- centre of mass: $$\left. \begin{aligned} \frac{1}{M} \int d^3R \, \underline{R} \, \rho(\underline{R}) &= \frac{1}{M} \int d^3R \, \underline{R} \sum_j m_j \, \delta(\underline{R} - \underline{R}_j) \\ &= \frac{1}{M} \sum_j m_j \underline{R}_j = \underline{R}_o \end{aligned} \right\} \qquad (4)$$

- Total momentum : $\underline{P} = \int d^3R\, \dot{\underline{R}}\, \rho(\underline{R}) = \int d^3R\, \dot{\underline{R}} \sum_j m_j\, \delta(\underline{R}-\underline{R}_j) = \sum_j m_j \dot{\underline{R}}_j$ (5)

Before continuing, we need to specify what makes a body rigid; the above eqtns would be valid for any set of particles or continuous medium, whether it be solid liquid, or gaseous. To define this, we first separate out the c.o.m. coordinate from the relative coordinates, writing.

$$\left.\begin{aligned} \underline{R} &= \underline{R}_o + \underline{r} & \dot{\underline{R}} &= \dot{\underline{R}}_o + \dot{\underline{r}} \\ \underline{R}_j &= \underline{R}_o + \underline{r}_j & \dot{\underline{R}}_j &= \dot{\underline{R}}_o + \underline{r}_j \end{aligned}\right\} \quad (6)$$

and so on. Now, for a body to be rigid, we require that

$$|\underline{r}_j(t)| = r_j \quad \text{const} \quad \forall\, j \qquad (7)$$

and this then implies that if the solid body is rotating at angular velocity $\underline{\omega}$ about some axis $\hat{\omega}$, then*

$$\dot{\underline{r}}_j(t) = \underline{\omega} \times \underline{r}_j(t) \quad ; \quad (|\underline{r}_j(t)| = r_j = \text{const.}) \qquad (8)$$

We can take this as the defining property of the solid body. Notice that, since $\underline{R}_o(t)$ is the centre of mass, we have

$$\sum_j m_j \underline{r}_j = \int d^3r\, \rho(r)\, \underline{r} = 0 \qquad (9)$$

We note that the total momentum of the system is just $M\dot{\underline{R}}_o$; for we have

$$\left.\begin{aligned} \underline{P} = \sum_j m_j \dot{\underline{R}}_j &= M\dot{\underline{R}}_o + \sum_j m_j (\underline{\omega} \times \dot{\underline{r}}_j) \\ &= M\dot{\underline{R}}_o \end{aligned}\right\} \qquad (10)$$

because $\sum_j m_j (\underline{\omega} \times \dot{\underline{r}}_j) = \underline{\omega} \times \sum_j m_j \dot{\underline{r}}_j = 0$ from (9); notice that this would also be true for a liquid or solid, since we did not, in getting (10), use the fact the $|\underline{r}_j(t)| = \text{const.}$

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* Another way to write the condition $|\underline{r}_j(t)| = r_j = \text{const}$ is to simply write $\underline{r}_j \cdot \dot{\underline{r}}_j = 0$

ENERGY & ANGULAR MOMENTUM: The key quantities one deals with in the rotation of a body (solid or liquid) are the energy E and angular momentum $\underline{L}$; both are conserved in the absence of external forces. Let us define them using the same routine as above.

Angular Momentum $\underline{L}$: In the inertial frame this is defined in the usual way as

$$\underline{L}(t) = \sum_j m_j (\underline{R}_j \times \dot{\underline{R}}_j) \equiv \int d^3R \, \rho(\underline{R}) \, (\underline{R} \times \dot{\underline{R}}) \qquad (11)$$

and if we substitute in (6), we find that

$$\left.\begin{aligned} \underline{L}(t) &= \underline{M}(\underline{R}_0 \times \dot{\underline{R}}_0) + \sum_j m_j (\underline{r}_j \times (\underline{\omega} \times \underline{r}_j)) \\ &\equiv M(\underline{R}_0 \times \dot{\underline{R}}_0) + \int d^3r \, \rho(\underline{r}) \, (\underline{r} \times (\underline{\omega} \times \underline{r})) \\ &\equiv \underline{L}_{com}(\underline{R}_0, \dot{\underline{R}}_0) + \underline{L}^{Rot} \end{aligned}\right\} \qquad (12)$$

since the cross-terms $\sum_j m_j \underline{r}_j \times \underline{R}_0$ and $\sum_j m_j (\underline{R}_0 \times (\underline{\omega} \times \underline{r}_j))$ are both zero, because of (9).

Eqtn. (12) shows that we can separate the centre of mass angular momentum about some inertial point from the rotational angular momentum about the centre of mass.

We can rewrite the rotational angular momentum, using the familiar vector identity*

$$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b}) \qquad (13)$$

so that

$$\left.\begin{aligned} \underline{L}^{Rot} &= \sum_j m_j (\underline{r}_j \times (\underline{\omega} \times \underline{r}_j)) \\ &= \sum_j m_j [\, r_j^2 \underline{\omega} - \underline{r}_j (\underline{\omega} \cdot \underline{r}_j)] \\ &\equiv \int d^3r \, \rho(\underline{r}) [\, r^2 \underline{\omega} - \underline{r}(\underline{\omega} \cdot \underline{r})]. \end{aligned}\right\} \qquad (14)$$

It is helpful to write this out in component notation. Before doing so it is useful to make sure you can do this fairly automatically, and also that you are familiar with the "Einstein summation convention", according to which any repeated indices are summed over — except in cases where it is specified

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* In class I showed you how one can get this identity without multiplying out all the components.

the summation is absent.*

To see how it works, consider how we write several simple vector identities in component notation:

$$\left.\begin{aligned} x &= \underline{A}\cdot\underline{B} &\equiv\quad x &= A_\alpha B_\alpha \\ \underline{a} &= \underline{A}\,(\underline{B}\cdot\underline{C}) &\equiv\quad a_\alpha &= A_\alpha B_\beta C_\beta \\ y &= (\underline{A}\cdot\underline{B})(\underline{C}\cdot\underline{D}) &\equiv\quad y &= A_\alpha B_\alpha C_\beta D_\beta \end{aligned}\right\} \quad (15)$$

Notice that we can rewrite all these identities in another form, involving multiplication of row and column vectors; thus, eg.,

$$x = A_\alpha B_\alpha \equiv (A_1 \; A_2 \; A_3)\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \quad (16)$$

and so on. We can continue this to look at multiplication of matrices, or matrices times vectors; eg., for 2 matrices $\underline{\underline{M}}$ and $\underline{\underline{N}}$, we have the multiplication

$$A_{\alpha\beta} = M_{\alpha\gamma} N_{\gamma\beta} \equiv \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}\begin{pmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{pmatrix} \quad (17)$$

and

$$A_\alpha = M_{\alpha\beta} B_\beta \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \quad (20)$$

and so on. These tensor relations lend go to higher rank tensors; eg., and eqtn. like $A_{\alpha\beta} = M_{\alpha\beta\gamma\delta} B_{\gamma\delta}$. Amongst these higher rank tensors there is the 3rd-rank tensor called the "unit antisymmetric tensor of 3rd rank", which is defined by**

$$\epsilon_{\alpha\beta\gamma} = \begin{cases} +1 \\ -1 \end{cases} \text{for } \alpha\beta\gamma = \begin{cases} \text{even permutation of } 123 \\ \text{odd permutation of } 123 \end{cases} \Bigg|\; 0 \text{ otherwise} \quad (21)$$

* Einstein invented this notation when developing the theory of General Relativity in tensor notation, to simplify the equations.

** The material here on 3rd-rank & higher tensors is for your interest only.

This tensor can be used to define vector cross products, as follows:

$$\left.\begin{aligned} \underline{b}\times\underline{c} &= \underline{d} &\equiv&\quad \epsilon_{\alpha\beta\gamma}\, b_\beta c_\gamma = d_\alpha \\ \underline{a}\cdot(\underline{b}\times\underline{c}) &= x &\equiv&\quad \epsilon_{\alpha\beta\gamma}\, a_\alpha b_\beta c_\gamma = x \end{aligned}\right\} \quad (22)$$

Having made this brief diversion, lets rewrite (14) in component notation - we have

$$\left.\begin{aligned} r^2\underline{\omega} - \underline{r}(\underline{\omega}\cdot\underline{r}) &= r^2\omega_\alpha - r_\alpha \omega_\beta r_\beta \\ &\equiv (r^2\delta_{\alpha\beta} - r_\alpha r_\beta)\,\omega_\beta \end{aligned}\right\} \quad (23)$$

where $\delta_{\alpha b}$ is the unit tensor, so that $\delta_{\alpha\beta}\omega_\beta = \omega_\alpha$. (since $\delta_{\alpha\beta} = 0$ unless $\alpha = \beta$).

This means we can also write (14) as

$$\underline{L}^{Rot} = \underline{\underline{I}}\cdot\underline{\omega} \equiv L_\alpha^{Rot} = I_{\alpha\beta}\,\omega_\beta \quad (24)$$

where the tensor $\underline{\underline{I}} \equiv I_{\alpha\beta}$ is given by

$$\left.\begin{aligned} I_{\alpha\beta} &= \int d^3r\,\rho(r)\left[r^2\delta_{\alpha\beta} - r_\alpha r_\beta\right] \\ &\equiv \sum_j m_j\left[r_j^2\delta_{\alpha\beta} - r_{j\alpha} r_{j\beta}\right] \end{aligned}\right\} \quad (25)$$

where $r_{j\alpha}$ is the α-th component of $\underline{r}_j$. This tensor is called the "inertia tensor", and we discuss its properties below.

<u>Rotational Energy</u> T_{Rot} : Consider now the total energy of the system. We ignore any external potential, & consider only the kinetic energy, which is

$$\left.\begin{aligned} T = \tfrac{1}{2}\sum_j m_j \dot{\underline{R}}_j^2 &\equiv \tfrac{1}{2}\sum_j \left(\dot{\underline{R}}_0 + (\underline{\omega}\times\underline{r}_j)\right)^2 \\ &= \tfrac{1}{2}\sum_j m_j \dot{\underline{R}}_0^2 + \tfrac{1}{2}\sum_j m_j (\underline{\omega}\times\underline{r}_j)^2 \end{aligned}\right\} \quad (26)$$

where the cross-term $\frac{1}{2}\sum_j m_j \left(2\dot{\underline{R}}_0 \cdot (\underline{w}\times \underline{r}_j)\right) = 0$ because $\sum_j m_j \underline{r}_j = 0$.
We rewrite (26) as

$$T = T_{com} + T^{Rot}. \tag{27}$$

with

$$\left.\begin{aligned} T_{com} &= \tfrac{1}{2} M \dot{\underline{R}}_0^2 \\ T^{Rot} &= \tfrac{1}{2}\sum_j m_j (\underline{\omega}\times \underline{r}_j)^2 \equiv \tfrac{1}{2}\int d^3_r\, \rho(\underline{r})\, (\underline{w}\times\underline{r})^2 \end{aligned}\right\} \tag{28}$$

Using the vector identity $(\underline{a}\times\underline{b})^2 = a^2b^2 - (\underline{a}\cdot\underline{b})^2$, we can rewrite eqtn (28) as:

$$T^{Rot} = \tfrac{1}{2}\sum_j m_j \left[r_j^2 w^2 - (\underline{r}_j\cdot\underline{w})^2\right] \equiv \tfrac{1}{2}\int d^3_r\, \rho(\underline{r}) \left[r^2 w^2 - (\underline{r}\cdot\underline{w})^2\right] \tag{29}$$

Again, if we write this in component form we attain some clarification; we have

$$\left.\begin{aligned} T_{Rot} &= \tfrac{1}{2}\sum_j m_j \left[r_j^2 w^2 - r_j^\alpha w^\alpha r_j^\beta w^\beta \right] \\ &\equiv \tfrac{1}{2}\sum_j m_j \left[r_j^2 \delta_{\alpha\beta} - r_{j\alpha} r_{j\beta}\right] \omega_\alpha w_\beta \\ &\equiv \tfrac{1}{2} I_{\alpha\beta} w_\alpha w_\beta \equiv \tfrac{1}{2}\underline{w}\cdot \underline{\underline{I}}\cdot\underline{w} \end{aligned}\right\} \tag{30}$$

Let's now summarize these relationships in a little table, comparing the rotational and translational motion of a solid body.

<u>TRANSLATIONAL MOTION</u>	<u>ROTATIONAL MOTION</u>
$\underline{P} = M\dot{\underline{R}}_0$	$\underline{L}^{Rot} = \underline{\underline{I}}\cdot\underline{w}$ ie. $L_\alpha^{Rot} = I_{\alpha\beta} w_\beta$
$T_{com} = \frac{1}{2}M\dot{\underline{R}}_0^2 = \frac{1}{2}P^2/M.$	$T^{Rot} = \frac{1}{2}\underline{w}\cdot\underline{\underline{I}}\cdot\underline{w} = \frac{1}{2}\underline{L}\cdot\underline{\underline{I}}^{-1}\cdot\underline{L}$ ie $T^{Rot} = \frac{1}{2} I_{\alpha\beta} w_\alpha w_\beta = \frac{1}{2} I^{-1}_{\alpha\beta} L_\alpha L_\beta$

(T.1)

Note that to get the final relations in this table, we use, from the first ones, that

$$\left.\begin{aligned} \dot{\underline{B}}_o &= M^{-1}\underline{P} \\ \underline{\omega} &= \underline{\underline{I}}^{-1}\cdot\underline{L} \qquad (\text{ie } \omega_\alpha = I^{-1}_{\alpha\beta} L_\beta) \end{aligned}\right\} \quad (31)$$

where the inverse matrix $I^{-1}_{\alpha\beta}$ is defined by

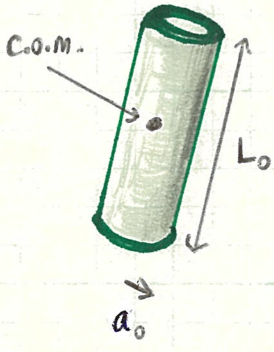
$$I^{-1}_{\alpha\gamma} I_{\gamma\beta} = \delta_{\alpha\beta} \quad (32)$$

in the usual way.

INERTIA TENSOR

The tensor $I_{\alpha\beta}$ we have just defined completely defines the dynamics of rotation for a given body (ie, its rotational properties). It is easiest to evaluate it about the centre of mass, and easier still to evaluate it in the "principal axis" reference frame - because $I_{\alpha\beta}$ is a real symmetric matrix, it can always be diagonalized by some rotation to a new coordinate frame. If the object has certain symmetries, these axes will also be axes of symmetry.

The easiest way to get a feel for this is via an example. We consider a solid cylinder, of uniform density ρ_o, and length L_o, radius a_o. Then the mass is



$$M = \pi a_o^2 L_o . \quad (33)$$

and we want to find $I_{\alpha\beta}$ about the centre of mass, situated on the cylinder axis, a distance $L_o/2$ from each end. We use eqtn (25) to do this; we then get the result

$$I_{\alpha\beta} = \rho_o \int d^3r \begin{pmatrix} r^2\sin^2\theta + z^2 & -r^2\cos\theta\sin\theta & -zr\cos\theta \\ -r^2\cos\theta\sin\theta & r^2\cos^2\theta + z^2 & -zr\sin\theta \\ -zr\cos\theta & -zr\sin\theta & r^2 \end{pmatrix} \quad (34)$$

where we choose cylindrical variables (z, r, θ), where r is the distance from the axis of symmetry.

You should now do the integrations in (34), using $\int d^3r = \int_0^{2\pi} d\theta \int_{-L_o/2}^{L_o/2} dz \int_0^{a_o} r\,dr$ to get the following result:

$$I_{\alpha\beta} = \begin{pmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{4}(a_0^4 L_0 + \frac{1}{3} a_0^2 L_0^3) & & 0 \\ & \frac{\pi}{4}(a_0^4 L_0 + \frac{1}{3} a_0^2 L_0^3) & \\ 0 & & \frac{\pi}{2} a_0^4 L_0 \end{pmatrix}$$

$$\equiv \begin{pmatrix} \frac{1}{4}M(a_0^2 + \frac{1}{3}L_0^2) & & 0 \\ & \frac{1}{4}M(a_0^2 + \frac{1}{3}L_0^2) & \\ 0 & & \frac{1}{2}Ma_0^2 \end{pmatrix} \qquad (35)$$

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Notice certain features of this result, viz.,

- We have chosen axes of symmetry, so the off-diagonal elements of $I_{\alpha\beta}$ vanish (here because all the angular integrals give zero).

- The moments of inertia I_1, I_2, I_3 along the principal axes are dominated by contributions far from these axes – this is because of the dependence on r^2 in (25). So these moments will be very small about axes for which the material of the body is close to these axes. Thus, eg., for a long thin cylinder, I_3 is very small, in (35), compared to I_1 and I_2; their ratio is $I_3/I_2 = I_3/I_1 \sim a_0^2/6L_0^2$.

- The direction of I_1 and I_2 in the xy-plane is arbitrary – they only have to be orthogonal. This is because the system is invariant under rotations about z.

EQUATION of MOTION UNDER TORQUE : After looking at the table T.1, the obvious question is: what is the analogue of Newton's 2nd law $\underline{F} = \dot{\underline{P}}$ for translational motion. The answer is of course that we have

$$\dot{\underline{L}}^{Rot} = \underline{\tau} \qquad (\text{ie., } \dot{L}_\alpha^{Rot} = \tau_\alpha) \qquad (36)$$

where $\underline{\tau}$ is the torque applied to the rotating body, defined in the usual way, viz.,

$$\underline{\tau} = \sum_j \underline{r}_j \times \underline{f}_j \equiv \frac{1}{M}\int d^3r\, \rho(\underline{r})\, (\underline{r} \times \underline{f}(\underline{r})) \qquad (37)$$

where $\underline{f}_j$ is the external force being applied to the j-th particle; if this force can be derived from a potential, ie $\underline{f}_j = -\partial V/\partial \underline{r}_j$, we can also see this.

(b) DYNAMICS IN THE ROTATING FRAME

As in our discussion of non-inertial motion, it is often more convenient to go to a frame rotating with the body itself - in this case, the inertia tensor does not change with time. It is then easy to transform the eqtn. of motion in (36) to this frame - we have

$$\underline{\tau} = \frac{\partial \underline{L}^{Rot}}{\partial t} + (\underline{\omega} \times \underline{L}(t)) \qquad \text{(rotating frame)} \qquad (38)$$

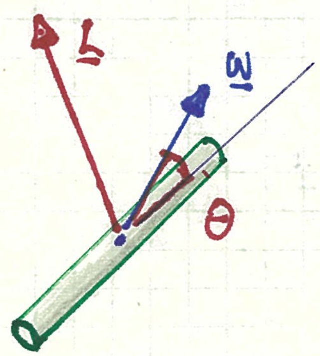
This eqtn is usually called the "Euler equation" (one of many named after him); when written out in components it takes the form (going here to the principal axis frame):

$$\left.\begin{aligned} \tau_1 &= I_1 \dot{\omega}_1 + (I_3 - I_2)\,\omega_2 \omega_3 \\ \tau_2 &= I_2 \dot{\omega}_2 + (I_1 - I_3)\,\omega_1 \omega_3 \\ \tau_3 &= I_3 \dot{\omega}_3 + (I_2 - I_1)\,\omega_1 \omega_2 \end{aligned}\right\} \qquad (39)$$

which constitutes a set of 3 coupled 1st-order non-linear differential eqtns - their general solutions are extremely complex.

To get some idea of how these solutions behave, let's look at some simple cases.

Example 1 : Rotating Rod : Let's go back to the rod we looked at earlier (with radius a_0 and length L_0). We will assume no torque - the rod is rotating freely. We consider the situation shown in the figure. We note that because $I_3 \ll I_1, I_2$, and since



$$\left.\begin{aligned} L_1 &= I_1 \omega_1 \\ L_2 &= I_2 \omega_2 \\ L_3 &= I_3 \omega_3 \end{aligned}\right\} \qquad (40)$$

that even though $\underline{\omega}$ is mostly directed along the z-axis, still $\underline{L}$ will be pointing in a quite different direction, much further from this axis, because L_3 will be much smaller that L_1 and L_2.

The dynamics here is simple to solve. Let's write that $I_1 = I_2 = I_\perp$, and then Euler's eqtns in (39) become

$$\left.\begin{aligned} I_\perp \dot{\omega}_1 - (I_\perp - I_3)\,\omega_2\omega_3 &= 0 \\ I_\perp \dot{\omega}_2 + (I_\perp - I_3)\,\omega_1\omega_3 &= 0 \\ I_3 \dot{\omega}_3 &= 0. \end{aligned}\right\} \quad (\text{EULER}) \qquad (41)$$

so that ω_3 is a constant. We then have a simple problem – the 1st 2 eqtns read

$$\left.\begin{aligned} \dot{\omega}_1 - \frac{(I_\perp - I_3)\,\omega_3}{I_\perp}\,\omega_2(t) &= 0. \\ \dot{\omega}_2 + \frac{(I_\perp - I_3)\,\omega_3}{I_\perp}\,\omega_1(t) &= 0 \end{aligned}\right\} \qquad (42)$$

and taking the time derivative of the 1st eqtn, we have, on substituting the 2nd eqtn, that

$$\ddot{\omega}_1 + \left(\frac{I_\perp - I_3}{I_\perp}\,\omega_3\right)^2 \omega_1(t) = 0 \qquad (43)$$

and the same eqtn for $\omega_2(t)$. We thus find oscillatory motion – since the 2 components will be $90°$ out of phase, we get

$$\left.\begin{aligned} \omega_1(t) &= \omega_1 \cos\left[\left(\left(\frac{I_\perp - I_3}{I_1}\right)\omega_3\right)t + \phi_0\right] \\ \omega_2(t) &= \omega_2 \sin\left[\left(\left(\frac{I_1 - I_3}{I_1}\right)\omega_3\right)t + \phi_0\right] \end{aligned}\right\} \qquad (44)$$

so that the axis $\underline{\omega}(t)$ remains at a fixed angle Θ w.r.t. the 2-axis, but rotates around it at a frequency $(I_\perp - I_3)\,\omega_3 / I_1$.

<u>EXAMPLE 2 : ASYMMETRIC ROD</u> : From the above, we see that the component of angular momentum about the 2-axis is a small perturbation in the dynamics. Let's now consider a system for which $I_1 \neq I_2 \neq I_3$, but still with the angular momentum $\underline{\omega}$ oriented nearly parallel to I_3. Let us solve for this assuming that $\omega_1, \omega_2 \ll \omega_3$, and write an "$\epsilon$-expansion" in the form:

$$\left.\begin{aligned} \omega_1 &= \sum_{n=1}^{\infty} \epsilon^n u_n = (\epsilon u_1 + \epsilon^2 u_2 + \dots) & \omega_3 &= \Omega_3 + \sum_{n=1}^{\infty} \epsilon^n \eta_n. \\ \omega_2 &= \sum_{n=1}^{\infty} \epsilon^n V_n = (\epsilon V_1 + \epsilon^2 V_2 + \dots) & &= (\Omega_3 + \epsilon\eta_1 + \epsilon^2\eta_2 + \dots) \end{aligned}\right\} \qquad (45)$$

where $\epsilon \ll 1$ is a small quantity. The situation here is exactly as was shown in our figure for the symmetric rod, with the axis of $\underline{\omega}$ very close to the cylinder axis, but now we are assuming that the rod cross-section is not symmetric with respect to rotations about the cylinder axis (the rod might now be elliptical in cross-section for example).

By introducing the ϵ factor, we are formalizing the idea that ω_1 and ω_2 are $\ll \omega_3$; we make them automatically smaller by a factor $\sim O(\epsilon)$. If $\epsilon \to 0$, then $\omega_1 = \omega_2 \equiv 0$, and $\omega_3 = \Omega_0$, and the problem is solved - we then get $\underline{L} \to \hat{e}_3 L_3$, and $L_1 = L_2 = 0$; and moreover, in the absence of external torque, we have

$$I_3 \dot{\omega}_3 \equiv I_3 \dot{\Omega}_3 = 0 \qquad (46)$$

so that $L_3 = I_3 \Omega_3$ for all time.

Now let's use the perturbation expansion (45) in the Euler equations. By substitution we get

$$\left.\begin{aligned}
I_1(\epsilon \dot{u}_1 + \epsilon^2 \dot{u}_2) + (I_3 - I_2)(\epsilon v_1 + \epsilon^2 v_2)(\Omega_3 + \epsilon \eta_1) &= 0 \\
I_2(\epsilon \dot{v}_1 + \epsilon^2 \dot{v}_2) + (I_1 - I_3)(\epsilon u_1 + \epsilon^2 u_2)(\Omega_3 + \epsilon \eta_1) &= 0 \\
I_3(\dot{\Omega}_3 + \epsilon \dot{\eta}_1 + \epsilon^2 \dot{\eta}_2) + (I_2 - I_1)(\epsilon u_1 + \epsilon^2 u_2)(\epsilon v_1 + \epsilon^2 v_2) &= 0
\end{aligned}\right\} \qquad (47)$$

We now want to find the solution to this - the equations have been written so that all terms up to $\sim O(\epsilon^2)$ are captured. Let's do this order by order:

ϵ^0 : Then only only one eqtn survives - we have

$$I_3 \dot{\Omega}_3 = 0. \implies \Omega_3 = \text{const} \qquad (48)$$

which is just eqtn (46) again.

ϵ^1 : We collect together terms $\sim O(\epsilon)$ to get

$$\left.\begin{aligned}
I_1 \dot{u}_1 + (I_3 - I_2)\Omega_3 v_1 &= 0 \\
I_2 \dot{v}_1 + (I_1 - I_3)\Omega_3 u_1 &= 0 \\
I_3 \eta_1 &= 0
\end{aligned}\right\} \qquad (49)$$

If we multiply these eqtns by ϵ, we see they have exactly the same structure as what we found for the symmetric rod in (41), except that in (41) we have $I_1 = I_2 = I_\perp$, whereas now they are different. However this difference does not yet show up in the last eqtn in (49), because there are no terms in eqtn (49) proportional to $(I_1 - I_2)$; if we look at the 3rd eqtn in (47), we see that the lowest-order terms proportional to $(I_1 - I_2)$ in this eqtn are $\sim O(\epsilon^2)$.

ϵ^2 : Collecting together terms $\sim O(\epsilon^2)$ from (47), we get:

$$\left.\begin{aligned} I_1 \dot{u}_2 + (I_3 - I_2)(v_1 \eta_1 + v_2 \Omega_3) &= 0 \\ I_2 \dot{v}_1 + (I_1 - I_3)(u_1 \eta_1 + u_2 \Omega_3) &= 0 \\ I_3 \dot{\eta}_2 + (I_2 - I_1) u_1 v_1 &= 0. \end{aligned}\right\} \qquad (50)$$

This is a complicated set of equations for the functions $u_2(t)$, $v_2(t)$, and $\eta_2(t)$, to be found in terms of the functions $\Omega_3 = \text{const}$, $u_1(t)$, $v_1(t)$, and $\eta_1(t)$ which we find from solving (48) and (49). Thus we see how one can acquire a systematic ϵ-expansion for $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$, by substituting the forms (45) into the Euler eqtns.