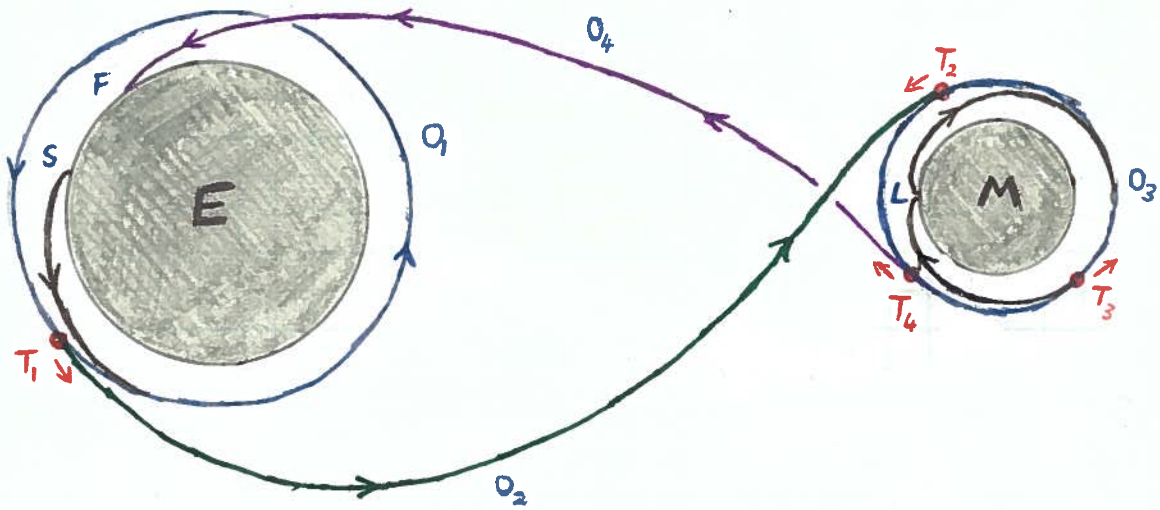
SOME OTHER ASPECTS OF CENTRAL FIELD MOTION

There are many other interesting features connected with motion in a central field - here we just look at a few. I will divide the discussion into questions about motion under gravity, taken to be Newtonian in form, and questions about motion in other kinds of central field.

ORBITAL TRANSFER : In the last 60 yrs we've all gotten used to the idea of spaceships climbing up from earth into various orbits, and shifting from one trajectory to another. These configurations can be fairly complicated, as we see from the following caricature of the earth-moon voyage :



I have considerably simplified things here from the real problem. The main features shown in the diagram are

(i) The spacecraft launches from S on earth, and attains an earth orbit O_1.

(ii) Later it applies thrust T_1, to propel it into another orbit O_2 heading for the moon.

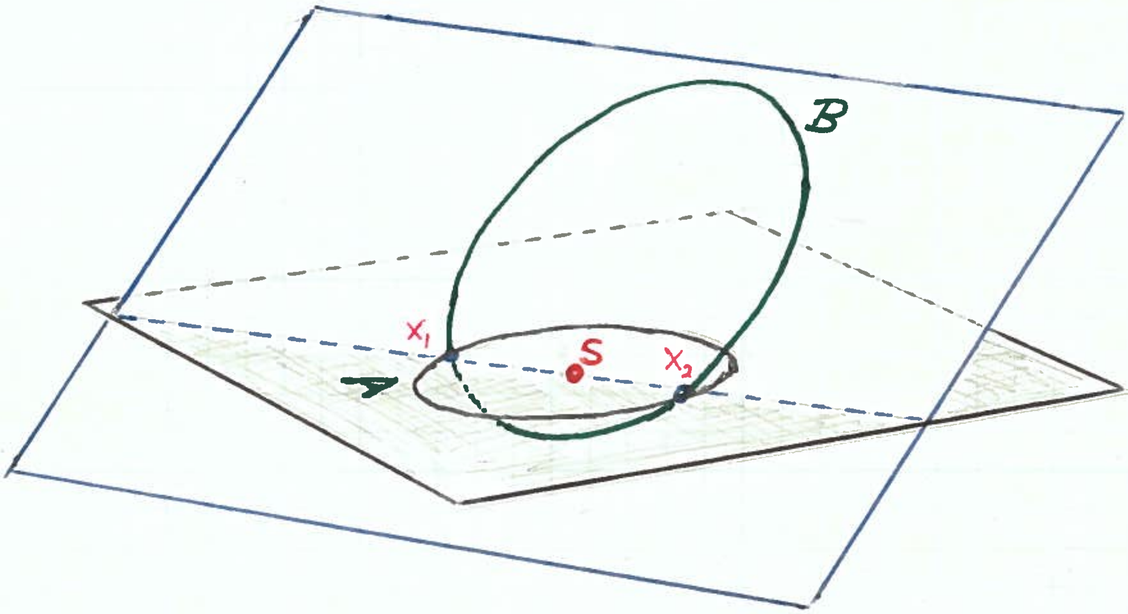
(iii) It then applies a braking thrust T_2, which puts it into orbit O_3 around the moon.

(iv) It applies a further braking thrust T_3, which drops it into an orbit taking it down to L on the moon's surface.

(v) After blasting off from the moon's surface to regain orbit O_3, it applies a final thrust T_4 to take it along orbit O_4 back to the earth.

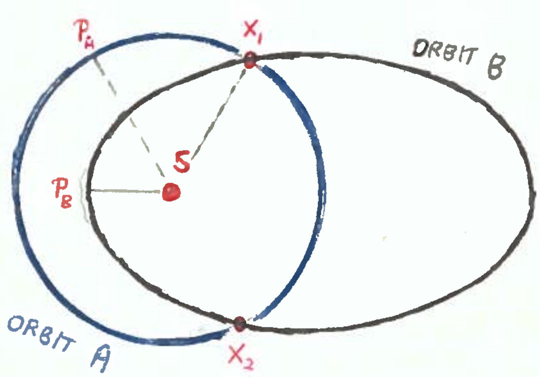
The real situation is more complex. Both the earth and moon are spinning, the earth has an atmosphere, the earth's spin axis is not perpendicular to the page, and both bodies have a gravitational field ; and neither is a perfect sphere, nor are they homogeneous.

However, we can make a good approximation. Apart from the thrusts applied to take off and land, we can treat the thrusts T_1, T_2, T_3, and T_4 as almost instantaneous, so that the orbits O_1, O_2, O_3, and O_4 are free orbits. This suggests another simpler problem, shown below:



We have 2 planes which intersect each other; the source of the central field is on the line of intersection between the planes, as are the 2 points of intersection of 2 orbits which lie on the 2 planes (points X_1 and X_2). Quite generally we can imagine a spacecraft starting in orbit A, and then applying a sudden thrust at X_1, which takes it into orbit B; it will recross orbit A at point X_2.

To analyze this problem is a little complicated, so we are going to simplify it by making the 2 orbits coplanar, to give the situation shown at below left. Now orbit A is almost circular, with its periastron at point P_1 (making an angle ϕ_A in the plane); and the strongly elliptical orbit B has periastron at point P_B. Again they cross at points X_1 and X_2. We can characterize each orbit by the equations



$$r_A(\phi) = \frac{R_A}{1 + e_A \cos(\phi - \phi_A)} \tag{1}$$

$$r_B(\phi) = \frac{R_B}{1 + e_B \cos(\phi - \phi_B)} \tag{2}$$

where R_A, R_B are the scale factors for each orbits, e_A, e_B are the eccentricities for the 2 orbits, and ϕ_A, ϕ_B the periastron angles. Finally, we introduce an angle Θ, which is the angle in the plane of the line connecting S with X_1.

We see that a typical situation will involve the application of a thrust to a rocket in orbit A, at point X_1, which sends it into orbit B. There are 2 obvious questions to answer here, viz

(i) What thrust do we have to apply in order to effect such a change in orbit (to make the required change in velocity).

(ii) If we know the change in velocity, how do we determine the parameters R_B, e_B, and ϕ_B, in terms of the parameters of the original orbit A, and knowing the angle Θ.

To solve this problem, note that there are 2 conserved quantities for each orbit, viz., ℓ_A and E_A for orbit A, and ℓ_B, E_B for orbit B; moreover, we know that

$$\left.\begin{aligned} R_\alpha &= \ell_\alpha^2 / mV_0 \qquad (3) \\ E_\alpha &= \frac{mV_0^2}{2\ell_\alpha^2}\left(e_\alpha^2 - 1\right) \qquad (4) \end{aligned}\right\} \quad \alpha = A \text{ or } B$$

from our previous work on elliptic orbits; we assume here a gravitational potential

$$V(r) = -V_0/r \tag{5}$$

from the source S. Thus we can find e_B and R_B if we know ℓ_B and E_B, and it is fairly clear that if we know the velocity of the spacecraft at point X_1 in orbit B, AND we know the angle Θ, then we can find the new orbit B. This is because

(a) knowledge of $\dot{\underline{r}}_B$ at point X_1 immediately tells us what is ℓ_B, since

$$\underline{\ell}_B = m(\dot{\underline{r}}_B \times \underline{r}_B) \tag{6}$$

(b) it then also tells us what is E_B, because

$$E_B = m\dot{r}_B^2 - \frac{V_0}{r_B} + \ell_B^2/2mr_B^2 \tag{7}$$

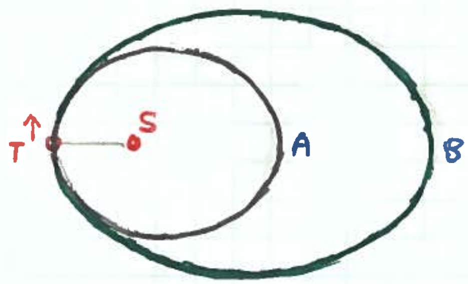
where we note that $\dot{r}_B$

Finally, we can determine the angle ϕ_B if we know Θ and ϕ_A, by simply equating (1) and (2) at the point X_1 where the orbits join, ie.,

$$R_A/[1 + e_A\cos(\Theta - \phi_A)] = R_B/[1 + e_B\cos(\Theta - \phi_B)] \tag{8}$$

Notice that we have described how the solution to our 2nd problem above, but not to the 1st one (of finding the thrust required to give us the desired new velocity $\dot{\underline{r}}_B$ at point X_1. To solve this we need to know the velocities $\dot{\underline{r}}_A(X_1)$ and $\dot{\underline{r}}_B(X_1)$, and then $\Delta \underline{V} = (\dot{\underline{r}}_B(X_1) - \dot{\underline{r}}_A(X_1))$. This means knowing the explicit solutions for the orbits, ie., knowing the solutions for $r_\alpha(t)$ and $\phi_\alpha(t)$ for the 2 orbits.

To see how all this works, we consider a very simple example. Suppose we have the situation shown in the figure, where orbit A and orbit B cross at a single point – for this to be the case, this point has to be either the periastron (point of closest approach) or apastron (point of furthest distance) from S, for each of the orbits. Here I show the case where this point (marked by T in the figure) is the periastron for both orbits.



It then follows that to transfer from orbit A to B (or vice-versa) we must apply thrust at this point; moreover, it must be applied in a direction parallel to the motion of the rocket, ie., a tangential thrust. This geometry is very simple, and may be solved immediately. The angle of periastron is taken to be $\phi_A = \phi_B = 0$, so that the 2 orbits have shapes given by

$$\left.\begin{aligned} r_A(\phi) &= \frac{R_A}{1 + e_A \cos\phi} = \frac{1}{mV_0} \frac{l_A^2}{1 + e_A \cos\phi} \\ r_B(\phi) &= \frac{R_B}{1 + e_B \cos\phi} = \frac{1}{mV_0} \frac{l_B^2}{1 + e_B \cos\phi} \end{aligned}\right\} \qquad (9)$$

But now since l_α at periastron is just proportional to $V_\alpha = |\dot{r}_\alpha|$, for $\alpha = A, B$ (in fact we have $l_\alpha = m r_{min} V_\alpha$, where r_{min} is the radius at periastron for both orbits), and since we also have $r_A(\phi = 0) = r_B(\phi = 0) = r_{min}$ for the 2 orbits, we can simply equate the 2 expressions in (9) for $\phi = 0$, ie., where the thrust is applied. We then have that

$$l_B^2 / l_A^2 = \frac{V_B^2}{V_A^2} = \frac{1 + e_B}{1 + e_A} \qquad (10)$$

Thus, if we increase the velocity at the periastron, using a thrust T, by a ratio $\gamma = V_B/V_A$, we will change the eccentricities according to

$$\gamma = \left(\frac{1 + e_B}{1 + e_A} \right)^{1/2} \qquad (11)$$