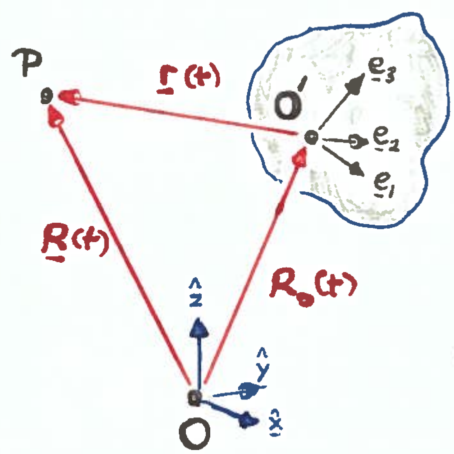
CLASSICAL MECHANICS IN NON-INERTIAL FRAMES

So far we've looked at classical dynamics in inertial frames, for good reason — they are simple in such frames. However in real life, we often work in non-inertial frames, because they are accelerating and/or rotating. The relationship between the two is thus of practical importance; and, as noted by Newton & Einstein, it is of fundamental theoretical importance as well.

In what follows we clarify what we mean by non-inertial frames, and then find the form of the Lagrangian & eqtns of motion in these frames. We then look at a few simple examples.

(a) TRANSFORMATION TO A NON-INERTIAL FRAME

: Let's begin by considering 2 frames, as shown in the figure. We begin by looking at how things relate to each other as viewed from the inertial frame O. We imagine looking at 2 points in space, at $\underline{R}_0(t)$ and $\underline{R}(t)$. The point at $\underline{R}_0(t)$ is the position from where the non-inertial frame is defined – we can imagine it fixed in some solid body, perhaps at its centre of mass, at O'. The second point at P can be thought of as defining the position of some particle. Its position relative to O is $\underline{R}(t)$, and relative to O' it is at $\underline{r}(t)$.



Now a key point here is that not only is O' moving with respect to O (at a velocity $\dot{\underline{R}}_0(t)$), but it also has a set of axes defining directions in O', defined by a set of orthonormal vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$; and there is no need for these axes to be aligned along the unit vectors $\hat{x}, \hat{y}, \hat{z}$ which define the inertial frame directions. The most general motion of the set $\{\hat{e}_j\}$ with respect to the inertial set $\hat{x}_j \equiv \{\hat{x}, \hat{y}, \hat{z}\}$ is a rotation; and it is easy to see that if the rotation has angular velocity $\underline{\omega}$ (ie, about an axis $\hat{\omega}$, with rate ω), then

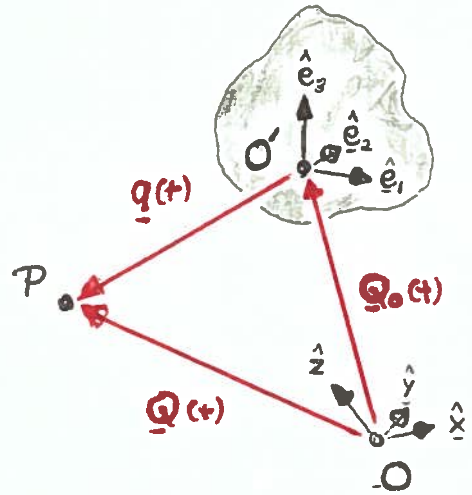
$$\dot{\hat{e}}_j = \frac{d}{dt}\hat{e}_j = \underline{\omega} \times \hat{e}_j \qquad (\text{in } O) \qquad (1)$$

Now we want to see how to define coordinates, velocities, and vectors in the non-inertial frame. Obviously in the inertial frame we have

$$\left.\begin{aligned} \underline{R}(t) &= \underline{R}_0(t) + \underline{r}(t) \\ \dot{\underline{R}}(t) &= \dot{\underline{R}}_0(t) + \dot{\underline{r}}(t) \end{aligned}\right\} \qquad (2)$$

and the properties of any other vector $\underline{A}(t)$ defined in this frame are simple enough.

Consider now how things look to an observer in the frame O', with its axes $\{\hat{e}_j(t)\}$ defined as above. It is important to see that the vectors defining the points P, O', and O are now different. Thus O' is now the origin, and there is a new time-dependence in three vectors coming from the rotation of the axes $\{\hat{e}_j(t)\}$. To see this in an extreme form, suppose that the vectors $\underline{R}$, $\underline{R}_o$, and $\underline{r}$ in the inertial frame are all fixed and independent of time. Nevertheless, if the frame attached to O' is rotating in the inertial frame, the corresponding vectors $\underline{Q}(t)$, $\underline{Q}_o$, and $\underline{q}(t)$ in the non-inertial frame will be time-dependent.



To see how this affects some vector as time goes on, let's consider a situation where at some given time we look at the time derivative of a vector $\underline{A}(t)$, written as

$$\underline{A}(t) = \begin{cases} \sum_j a_j(t)\,\hat{x}_j + \underline{A}_O & (\text{in } O) \\ \sum_j a_j(t)\,\underline{e}_j(t) + \underline{A}_{O'} & (\text{in } O') \end{cases} \qquad (3)$$

where to simplify what follows, we have lined up the axes $\hat{e}_j(t)$ and $\hat{x}_j$ to be parallel at time t (if we relax this restriction, we simply complicate the algebra, but don't change the answer); $\underline{A}_O$ and $\underline{A}_{O'}$ are two vectors which make no difference to the result that follows below*

Now consider the time differential of $\underline{A}(t)$; we have

$$\dot{\underline{A}}(t) = \sum_j \dot{a}_j(t)\,\hat{x}_j = \dot{\underline{a}}(t) \qquad (\text{in } O) \qquad (4)$$

$$\left.\begin{aligned} &\equiv \sum_j \left[\dot{a}_j(t)\,\underline{e}_j(t) + a_j(t)\,\dot{\underline{e}}_j(t)\right] \\ &\equiv \dot{\underline{a}}(t) + (\underline{\omega} \times \underline{A}(t)) \end{aligned}\right\} \quad (\text{in } O') \qquad (5)$$

so that in the rotating frame, we have an extra contribution to the derivative coming

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* These vectors A_O and $A_{O'}$ simply correspond to a shift of origin in the 2 systems, and we will ignore them from now on. One can write (4) and (5) in various ways; eg., $\dot{A}(t) = \partial A/\partial t + (\omega \times A)$.

from the rotation of the frame of reference centred at O'. Notice, incidentally, that one vector that is the same in the 2 frames of reference is $\underline{\omega}$; this is because $\underline{\omega} \times \underline{\omega} = 0$.

We now apply this result to find the eqtn. of motion of a particle P in the non-inertial frame.

(b) LAGRANGIAN & EQTNS of MOTION IN NON-INERTIAL FRAME

To derive the forms for $\mathcal{L}$, the Lagrangian, and the eqtns of motion in the non-inertial frame, we will use the transformations derived above. To do so we consider the simple Lagrangian

$$\mathcal{L} = \tfrac{1}{2} m \dot{\underline{R}}^2 - V(\underline{R}) \tag{6}$$

for the particle at P. In what follows we look first at how the Lagrangian transforms under the transformation to O' from O, and then how the eqtns. of motion transform.

TRANSFORMED LAGRANGIAN: Suppose we consider first, for simplicity's sake, the case where $\omega = 0$, ie., the frame at O' is not rotating. Then we have the simple substitution in (2), ie.,

$$\begin{aligned} \mathcal{L}' &= \tfrac{1}{2} m (\dot{\underline{R}}_0 + \dot{\underline{r}})^2 - V(\underline{R}_0 + \underline{r}) \\ &= \tfrac{1}{2} m (\dot{\underline{R}}_0^2 + \dot{\underline{r}}^2) + m \dot{\underline{R}}_0 \cdot \dot{\underline{r}} - V(\underline{R}_0 + \underline{r}) \end{aligned} \tag{7}$$

Now we can get rid of 2 of the terms here by noting that the equations of motion and the Lagrangian are unaffected (or only affected trivially) by adding any term to the Lagrangian which is a total time derivative*. There are actually two total time derivatives concealed in (7); we note that the scalar quantity $\tfrac{1}{2} m \dot{R}_0^2$ is clearly the time derivative of its integral, which is a function only of R_0 and t; and the function $m \dot{\underline{R}}_0 \cdot \dot{\underline{r}}$ can be written as

$$m \dot{\underline{R}}_0 \cdot \dot{\underline{r}} = m \left[\frac{d}{dt}(\dot{\underline{R}}_0 \cdot \underline{r}) - \ddot{\underline{R}}_0 \cdot \underline{r} \right] = m \frac{d}{dt}(\dot{\underline{R}}_0 \cdot \underline{r}) - \underline{F}_{inertial} \cdot \underline{r} \tag{8}$$

so that we can write

$$\mathcal{L}' = \tfrac{1}{2} m \dot{r}^2 - \underline{F}_{inert.} \cdot \underline{r} - V(\underline{r}) + T.T.D. \tag{9}$$

where "T.T.D." means "total time derivative", and we have written $V(\underline{r})$ in place of $V(\underline{R} + \underline{r})$ because our new origin is at O'. The "force" $\underline{F}_{inert}$ is an "inertial

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* Suppose we add a term $f(\varphi, t) = \frac{d}{dt} F(\varphi, t)$ to $\mathcal{L}_0(\varphi, \dot{\varphi}; t)$. Then the new action is $S = \int_{t_1}^{t_2} dt\, [\mathcal{L} + dF/dt] = S_0 + (F(t_2, \varphi) - F(t_1, \varphi))$ which just adds a constant to S_0; this cannot affect the eqtns of motion.

force"; we have

$$\underline{F}_{inert} = m\ddot{\underline{R}}_0 \qquad (10)$$

and it comes from the acceleration of O′. Thus in the frame O′, provided it is not rotating, we will feel an extra force $\underline{F}_{inert}$ coming from the acceleration of O′. This eqtn. expresses very succinctly, in the framework of Newtonian mechanics, the "equivalence principle", viz., that we cannot, in some reference frame, distinguish between the effect of some external force, and the effect of an acceleration of the reference frame. When generalized to relativistic dynamics, this result becomes the "Einstein equivalence principle."

Now suppose we allow rotations in the frame of reference centred at O′. Then, according to (5), we get instead of (9) the transformed Lagrangian

$$\mathcal{L}' = \tfrac{1}{2}m(\dot{\underline{r}} + (\underline{\omega}\times\underline{r}))^2 - \underline{F}_{inert}\cdot\underline{r} - V(\underline{r}) \qquad (11)$$

where I already dealt with the total time derivatives in the same way as above; multiplying this out we get

$$\boxed{\mathcal{L}' = \tfrac{1}{2}m\dot{\underline{r}}^2 + \underbrace{\tfrac{1}{2}m(\underline{\omega}\times\underline{r})^2}_{\text{centrifugal potential}} + \underbrace{m\dot{\underline{r}}\cdot(\underline{\omega}\times\underline{r})}_{\text{Coriolis potential}} - \underbrace{\underline{F}_{inert}\cdot\dot{\underline{r}}}_{\text{inertial term}} - V(\underline{r})} \qquad (12)$$

with the names sometimes given to these different terms.

<u>TRANSFORMED EQTNS of MOTION</u>: We can if we like derive the eqtns of motion directly from (12) using Lagrange's eqtns. But it is just as easy, if not easier, to just derive them by transforming Newton's 2nd law from O to O′, again by substitution into

$$M\ddot{\underline{R}} + \partial V/\partial \underline{R} = 0 \qquad (13)$$

Going over to the O′ frame, and noting that $\ddot{\underline{R}} = \ddot{\underline{R}}_0 + \frac{d^2}{dt^2}\underline{r}(t)$, we have

$$M\left(\ddot{\underline{R}}_0 + \frac{d^2}{dt^2}\underline{r}(t)\right) + \partial V/\partial \underline{r} = 0 \qquad (14)$$

Now let us work out the 2nd term. We have, in the rotating frame,

$$\left.\begin{aligned}\frac{d^2}{dt^2}\underline{r}(t) &= \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) = \frac{d}{dt}\left(\dot{\underline{r}} + (\underline{\omega}\times\underline{r})\right)\\ &= \left(\frac{d\dot{\underline{r}}}{dt}\right) + \frac{d}{dt}(\underline{\omega}\times\underline{r})\\ &= \left(\ddot{\underline{r}} + (\underline{\omega}\times\dot{\underline{r}})\right) + \left[\left(\frac{d\underline{\omega}}{dt}\times\underline{r}\right) + \left(\underline{\omega}\times\frac{d}{dt}\underline{r}\right)\right]\end{aligned}\right\} \qquad (15)$$

and then, noting that $d\underline{\omega}/dt = \dot{\underline{\omega}}$ because $\underline{\omega} \times \underline{\omega} = 0$ (cf. the remarks just after eqtn (5)), we finally get, in the rotating O′ frame, that

$$\left.\begin{aligned} \frac{d^2}{dt^2}\underline{r}(t) &= \ddot{\underline{r}} + \underline{\omega}\times\dot{\underline{r}} + \dot{\underline{\omega}}\times\underline{r} + \underline{\omega}\times[\dot{\underline{r}} + (\underline{\omega}\times\underline{r})] \\ &= \ddot{\underline{r}} + 2(\underline{\omega}\times\dot{\underline{r}}) + \dot{\underline{\omega}}\times\underline{r} + \underline{\omega}\times(\underline{\omega}\times\underline{r}) \end{aligned}\right\} \qquad (16)$$

so that the eqtn of motion in (13) becomes, from (14) and (15),

$$\boxed{m\ddot{\underline{r}} + \underbrace{2m(\underline{\omega}\times\dot{\underline{r}})}_{\text{Coriolis}} + \underbrace{m(\dot{\underline{\omega}}\times\underline{r})}_{\text{"Wobble"}} + \underbrace{m\,\underline{\omega}\times(\underline{\omega}\times\underline{r})}_{\text{Centrifugal}} = \underline{F}_{Tot}} \qquad (17)$$

with a total force $\underline{F}_{Tot}$:

$$\underline{F}_{Tot} = \underline{F}_{inert} - \partial V/\partial \underline{r} \qquad (18)$$

with $\underline{F}_{inert}$ given by (10) above. The centrifugal & Coriolis forces are discussed below; the "wobble" force comes from the fact that $\underline{\omega}$ itself may vary in time (eg., in a spacecraft, $\underline{\omega}$ is generally dependent on t, because of all the different thrusts being applied to it). We will not look at this wobble term here.

(b) CENTRIFUGAL & CORIOLIS FORCES

: Let's briefly look at how these work. We can most easily do so by considering examples – the simplest is to look at them separately; although clearly in any real situation they work together.

CENTRIFUGAL FORCE

: This is by far the most intuitively familiar to ordinary experience, mainly because it acts even when the velocity $\dot{\underline{r}} = 0$. We can write it as

$$\left.\begin{aligned} \underline{F}_{centrif} &= -m\,\underline{\omega}\times(\underline{\omega}\times\underline{r}) \\ &= -m[\underline{\omega}(\underline{\omega}\cdot\underline{r}) - \omega^2\underline{r}] \end{aligned}\right\} \qquad (19)$$

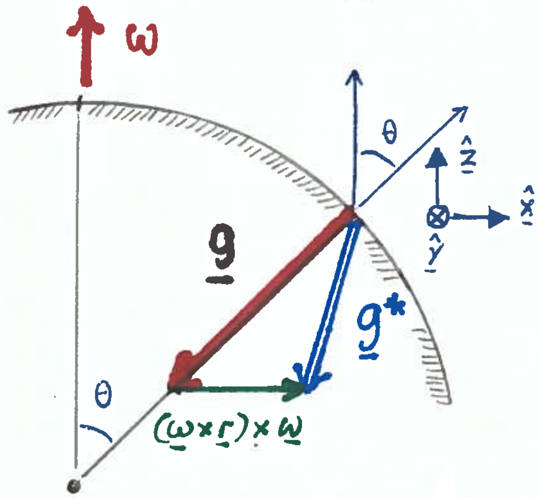
using the standard vector identity for ~~the~~ vector product $\underline{a}\times(\underline{b}\times\underline{c})$; in cylindrical coordinates with $\underline{\omega} = \hat{z}\omega$, this just becomes

$$\left.\begin{aligned} \underline{F}_{centrif} &= -m[(\omega r_z)\hat{z}\omega - \omega^2[\hat{z}r_z + \underline{r}_\perp]] \\ &= m\omega^2\underline{r}_\perp \end{aligned}\right\} \qquad (20)$$

where $\underline{r}_\perp = (\hat{\underline{x}} r_x + \hat{\underline{y}} r_y)$ is the radius vector perpendicular to the $\hat{\underline{z}}$ axis, ie., that component of $\underline{r}$ in the xy-plane.

It is interesting to see how this force affects our notion of "up" and "down" on the surface of the earth. The simplest way to see this is to imagine a spherical earth rotating at angular velocity ω, and see how the force acting on a stationary object deviates from the direction towards the centre of the earth.

The situation is that shown in the figure. We imagine a point on the earth at polar angle θ away* from the north pole, and so the forces on it come from 2 sources:



(i) There is a force mg, acting on a mass m, from the earth's gravity - this is directed towards the centre of the earth.

(ii) There is a centrifugal force directed perpendicular to $\underline{\omega}$, of magnitude

$$\underline{F}_{centrif} = m(\underline{\omega} \times \underline{r}) \times \underline{\omega} \tag{21}$$

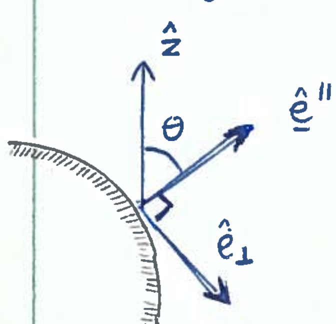
Thus the combined field acting on the mass m can be written as

$$\underline{g}^* = \underline{g} + (\underline{\omega} \times \underline{r}) \times \underline{\omega} = \hat{\underline{z}}\, g_*^z + \hat{\underline{x}}\, g_* \tag{22}$$

where in this case $\underline{r} = \hat{\underline{z}} r_z + \hat{\underline{x}} r_x$, and we have chosen the $\hat{\underline{x}}, \hat{\underline{y}}, \hat{\underline{z}}$ axes so that the y-axis points into the page; the components of $\underline{g}_*$ are then

$$\left.\begin{aligned} g_*^z &\quad -g\cos\theta \\ g_*^x &= -g\sin\theta + \omega^2 r_x \equiv (-g + R_\odot \omega^2)\sin\theta \end{aligned}\right\} \tag{23}$$

where $R_\odot$ is the radius of the earth. We can also write the answer using the axes shown below left; here $\hat{\underline{e}}^{\parallel}$ is a unit vector directed out from the centre of the earth, so $\hat{e}_\parallel \equiv \hat{\underline{r}}$, and $\hat{e}_\perp$ is parallel to the earth's surface, ie., horizontal at the local position on the earth's surface. We then write the vector g_* in this new coordinate system (which is the one that



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* In other words, the latitude is $\pi - \theta$.

we ourselves see) as

$$g_* = g_*^{\parallel}\,\underline{\hat{e}}^{\parallel} + g_*^{\perp}\,\underline{\hat{e}}^{\perp} \qquad (24)$$

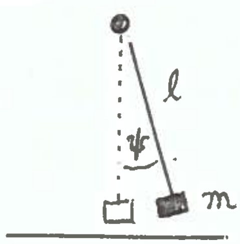
and we find that

$$\left.\begin{aligned} g_*^{\parallel} &= -g(1-\omega^2 R_\odot \sin^2\theta) \\ g_*^{\perp} &= \omega^2 R_\odot \sin\theta\cos\theta \end{aligned}\right\} \qquad (25)$$

Thus, as a result of the centrifugal force, we expect to see the weight of the mass m decrease; we will measure an APPARENT MASS

$$m^* = m(1-\omega^2 R_\odot \sin^2\theta) \qquad (26)$$

and we also see a slight sideways force $m g_*^{\perp}\,\hat{\underline{e}}_{\perp}$. We could measure the ratio of these 2 forces by hanging a weight; the angle ψ it then deviates from the vertical would be given by



$$\left.\begin{aligned} \tan\psi &= \left|\frac{g_*^{\perp}}{g_*^{\parallel}}\right| = \frac{\omega^2 R_\odot \sin\theta\cos\theta}{g(1-\omega^2 R_\odot\sin^2\theta)} \\ &\xrightarrow[\psi \ll 1]{} \frac{\omega^2 R_\odot}{g}\sin\theta\cos\theta \end{aligned}\right\} \qquad (27)$$

Now, even though $R_\odot$ is quite big for the earth, ω is also very small; we actually find that for the earth, $\omega^2 R_\odot \sim 3.4$ cm/sec^2, whereas $g \sim 9.81$ m/sec^2, so that the angle ψ is

$$\left.\begin{aligned} \psi &\sim 3.4\times 10^{-3}\sin\theta\cos\theta \\ &\sim 12'\sin\theta\cos\theta \end{aligned}\right\} \qquad (28)$$

for the earth, where $R_\odot \sim 6.37\times 10^3$ km, and $\omega = 7.292\times 10^{-5}\,s^{-1}$. Thus the deviation from the vertical for the earth is rather small, with a maximum when $\theta = 45°$, when $\psi \sim 6'$ of arc.*

We notice that for a more rapidly spinning body the effect of centrifugal force gets much bigger. However this centrifugal force also causes a large body, like a planet or star, to flatten, and if it is strong enough, the system will fly apart.

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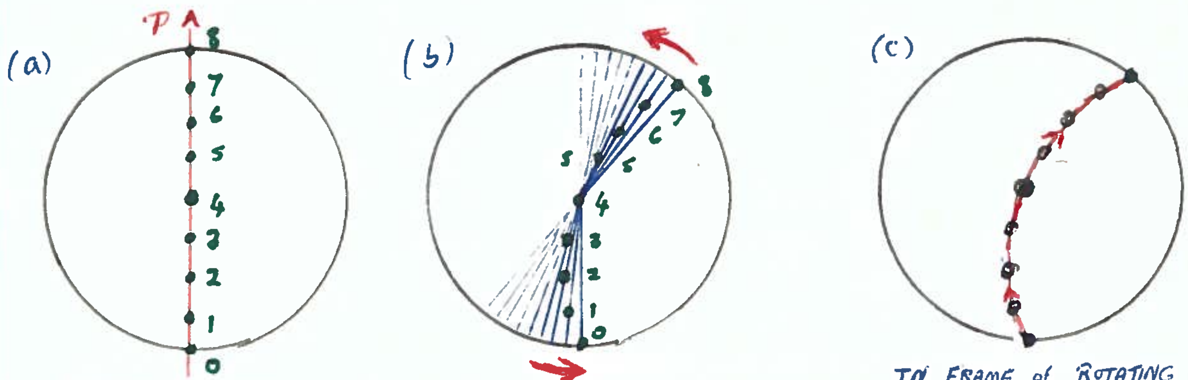
* In reality the earth is not a sphere; the deformation created by rotation causes a a flattening of roughly 1/300 from spherical, and this effect is comparable to the centrifugal contribution (and in the same direction).

CORIOLIS FORCE: Suppose that instead of hanging an object from some point (which, as we just saw, allows us in principle to measure the centrifugal force caused by earth's rotation) we actually drop the object.

At this point the Coriolis force comes into play. We see from eqtn (17) that faster it falls, the stronger is this force, and it is perpendicular to both $\dot{\underline{r}}$ and $\underline{\omega}$.



INERTIAL FRAME

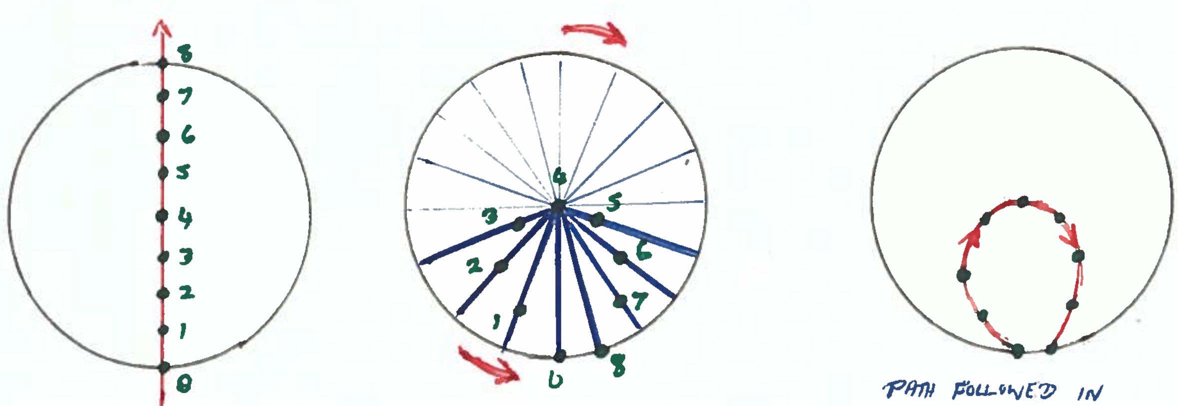
IN FRAME of ROTATING OBJECT

Above we show how this can work in practise. In (a), we show things in the inertial frame, with a non-rotating disc being traversed by an object following the path P (if you want, you can think of a fly passing over a rotating record, or a satellite on a circumpolar flight passing over the earth).

The second figure (b) shows the situation when the disc is rotating slowly ANTI-clockwise, as viewed from above (this corresponds to the earth's rotation, viewed from above the north pole). We show the way in which points on the moving disc slowly move under the path of the moving object to coincide with it as it passes. Thus, initially, points that were previously to "the left" of the straight line path must move into it in order to coincide with the moving object as it passes.

Finally, in (c), we show the set of points on the rotating object that end up coinciding with the object flying overhead. If this was the rotating earth, these are the points where observers would see the satellite flying overhead.

The situation is even more dramatic if the rotating disc/earth is rotating faster: the path as viewed in the rotating frame curls up more & more.



PATH FOLLOWED IN INERTIAL FRAME

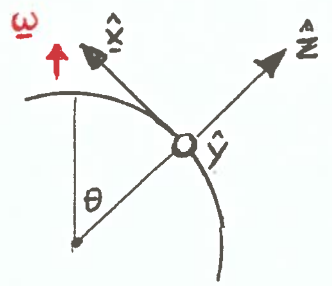
COINCIDING POINTS ON ROTATING OBJECT

PATH FOLLOWED IN FRAME OF ROTATING OBJECT

With these preliminary remarks in mind (which help us to see the direction of the force using purely physical arguments), let's now analyze an example. We have a Coriolis force given by

$$\underline{F}_{cor} = -2m(\underline{\omega} \times \dot{\underline{r}}) \tag{29}$$

and now we want to consider what happens if we drop an object from some height on a rotating body like the earth. The geometry is shown in the figure at left. We are at a point on the earth at a polar angle θ (i.e., a latitude $\pi - \theta$). The local axes are as shown; the $\hat{\underline{y}}$ axis is pointing out of the paper.



We now drop an object from a height z_0 above the ground – in what follows we will ignore centrifugal forces, so the total force on the object leads to an eqtn of motion

$$m\ddot{\underline{r}} = \underline{g}m - 2m(\underline{\omega} \times \dot{\underline{r}}) \tag{30}$$

Now to solve the trajectory for arbitrary heights and values of $\underline{\omega}$ and z_0 is actually quite messy, so we are going to assume that during the fall, the Coriolis force is a SMALL PERTURBATION on the gravitational force. To do this we rewrite (30) in the form

$$\ddot{\underline{r}} = \underline{g} - m\epsilon(\underline{\Omega} \times \dot{\underline{r}}) \qquad (\epsilon \ll 1) \tag{31}$$

where $\underline{\Omega}$ is not assumed small, $\underline{\omega} = \epsilon\underline{\Omega}$, and we write $\underline{r}(t)$ as

$$\underline{r}(t) = \underline{r}_0(t) + \epsilon \underline{r}_1(t) + \epsilon^2 \underline{r}_2(t) + \cdots \tag{32}$$

where $\underline{r}_0(t)$ is the "unperturbed solution", with $\epsilon = 0$, i.e., it obeys

$$\ddot{\underline{r}}_0(t) = \underline{g} \tag{33}$$

and the corrections to $\underline{r}_0(t)$ in (32) come from the Coriolis perturbation. Now, we will obtain a solution as an expansion in powers of ϵ. To do this we substitute the "ansatz" in (32) into the equation of motion, to get

$$(\ddot{\underline{r}}_0 + \epsilon\ddot{\underline{r}}_1 + \epsilon^2\ddot{\underline{r}}_2 + \cdots) = \underline{g} - 2\underline{\Omega} \times (\epsilon\dot{\underline{r}}_0 + \epsilon^2\dot{\underline{r}}_1 + \cdots) \tag{34}$$

and then, since ϵ is an arbitrary number which we can vary (always assuming that $\epsilon \ll 1$), we can equate powers of ϵ. This then gives us the eqtns.

$$\left.\begin{aligned} \epsilon^0 &: \quad \ddot{\underline{r}}_0 = \underline{g} \\ \epsilon^1 &: \quad \ddot{\underline{r}}_1 = -2(\underline{\Omega} \times \dot{\underline{r}}_0) \\ \epsilon^2 &: \quad \ddot{\underline{r}}_2 = -2(\underline{\Omega} \times \dot{\underline{r}}_1) \end{aligned}\right\} \qquad (35)$$

and so on. To find the solution we now integrate these eqtns up; notice that the solution for $\underline{r}_j(t)$, the term of order ϵ^j, depends on the solution for $\underline{r}_{j-1}(t)$, and so on - we have a coupled set of equations.

Solving the lowest-order eqtn for $\ddot{\underline{r}}_0$, with initial conditions $\underline{r}_0(t=0) = \hat{\underline{z}} z_0$, $\dot{\underline{r}}_0(t=0) = 0$, & writing $\underline{g} = -g\hat{\underline{z}}$, we find

$$\left.\begin{aligned} \underline{r}_0(t) &= \hat{\underline{z}}(z_0 - \tfrac{1}{2} g t^2) \\ \dot{\underline{r}}_0(t) &= -\hat{\underline{z}} g t \end{aligned}\right\} \qquad (36)$$

Note there is no need for us to write out the vector components here - if we just keep everything in vector notation we have

$$\left.\begin{aligned} \dot{\underline{r}}_0(t) &= \underline{g} t \\ \underline{r}_0(t) &= \underline{r}_0(t=0) + \tfrac{1}{2} \underline{g} t^2 \end{aligned}\right\} \qquad (37)$$

We now go to the eqtn of order ϵ in (35), which reads

$$\ddot{\underline{r}}_1 = -2(\underline{\Omega} \times \dot{\underline{r}}_0) = -2(\underline{\Omega} \times \underline{g}) t \qquad (38)$$

which gives

$$\left.\begin{aligned} \dot{\underline{r}}_1(t) &= \dot{\underline{r}}_1(0) - (\underline{\Omega} \times \underline{g}) t^2 = -(\underline{\Omega} \times \underline{g}) t^2 \\ \underline{r}_1(t) &= \underline{r}_1(0) - \tfrac{1}{3}(\underline{\Omega} \times \underline{g}) t^3 = -\tfrac{1}{3}(\underline{\Omega} \times \underline{g}) t^3 \end{aligned}\right\} \qquad (39)$$

where we use $\underline{r}_1(0) = \dot{\underline{r}}_1(0) = 0$; the initial conditions are all contained in $\underline{r}_0(0)$.
Now putting this all together, we get

$$\underline{r}(t) = \underline{r}_0(t) + \epsilon \underline{r}_1(t) = \underline{r}_0(0) + \tfrac{1}{2} \underline{g} t^2 - \tfrac{1}{3}(\underline{\omega} \times \underline{g}) t^3$$

for the solution up to order ϵ (here we have resubstituted $\underline{\omega} = \epsilon \underline{\Omega}$). If we write this out in components, noting that

$$\underline{\omega} = \omega [\hat{\underline{z}} \cos\theta + \hat{\underline{x}} \sin\theta] \qquad (40)$$

we then find that $\quad \underline{r}(t) = \hat{\underline{z}}(z_0 - \tfrac{1}{2} g t^2) - \hat{\underline{y}} \dfrac{\omega g t^3}{3} \sin\theta \qquad (41)$

Now the deflection here is pretty small if we choose appropriate values for the earth. From (41) we see that the time for the system to fall from a height z_0 is given from (36) by

$$t_0 = (2z_0/g)^{1/2} \qquad (42)$$

and we then get a deflection

$$y_0 = y(t=t_0) = \frac{1}{3}\omega g\left(\frac{2z_0}{g}\right)^{3/2}\sin\theta \qquad (43)$$

If we put the numbers in for the earth, viz.,

$$\left.\begin{aligned} \omega &= 7.292\times 10^{-5}\ \text{rads/sec} \\ g &= 9.81\ \text{ms}^{-2} \end{aligned}\right\} \qquad (44)$$

and go to $\theta = 45^\circ$ (ie, a latitude of 45°), and then drop an object from a height of 100 m, we find $y_0 \sim 1.5$ cm. This only increases like the 3/2 power of z_0; for a height of 1 km the deflection is 61 cm, and for a height of 10 km (an airplane altitude), we get a deflection ~ 15.5 m.

One can, if the perturbation is larger, continue the expansion in ϵ to higher order in ϵ; to amuse yourself, you could try solving up to $\sim O(\epsilon^2)$, using the 3rd eqtn in (35).