CENTRAL FIELD MOTION : BRIEF NOTES

Since most of the following material is fairly thoroughly discussed in the course text (see pp. 293-327, ie., Ch.8, of the course text), I will only give a brief discussion.

We are interested in the dynamics of a particle in a central field, so that the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}m\dot{\underline{r}}^2 - V(r) \qquad (1)$$

so the potential $V(r)$ is independent of angle ϕ.
We can write this in cylindrical coordinates, ie., $\underline{r} = (z, r, \phi)$, as

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) \qquad (2)$$

and, since we choose the orientation of the angular momentum $\underline{L}$ to be along $\hat{z}$, the Lagrangian is independent of z and $\dot{z}$. The angular momentum is

$$\underline{L} = \hat{z}L = \hat{z}\, mr^2\dot{\phi} \qquad (3)$$

and it is conserved. The other obvious conserved quantity is the total energy, given by

$$E = \tfrac{1}{2}m\dot{\underline{r}}^2 + V(r) \qquad (4)$$

Using (3), we can then write

$$\left.\begin{aligned} &(a) \quad \mathcal{L} = \tfrac{1}{2}m\dot{r}^2 + L^2/2mr^2 - V(r) \\ &(b) \quad E = \tfrac{1}{2}m\dot{r}^2 + \left(L^2/2mr^2 + V(r)\right) \end{aligned}\right\} \qquad (5)$$

In what follows we will define an effective radial potential $U_R(r)$ by the sum of the actual radial potential $V(r)$ and the centrifugal potential $L^2/2mr^2$, ie.,

$$U_R(r) = L^2/2mr^2 + V(r) \qquad (6)$$

Note that it would be incorrect to substitute $U_R(r)$ back into the Lagrangian, and write $\mathcal{L} = \tfrac{1}{2}m\dot{r}^2 - U_R(r)$; this contradicts (5(a)). So we have to be a little careful in thinking of $U_R(r)$ as a potential. However we do have

$$E = \tfrac{1}{2}m\dot{r}^2 + U_R(r) \qquad (7)$$

Now let's consider the eqtns. of motion for this system. We begin with the Lagrangian, written in the form (2); we then have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = m\ddot r - \frac{\partial}{\partial r}\left(\frac{m}{2} r^2 \dot\phi^2 - V(r)\right)$$

$$= m\ddot r - \left(m r \dot\phi^2 - \partial V/\partial r\right) = 0 \qquad (8)$$

which we write as

$$\boxed{m\ddot r = -\frac{\partial U_R(r)}{\partial r}} \qquad (9)$$

using (3) and (6). Note we would not have gotten this result if we had used the form (5(a)) for L; we need to apply Lagrange's eqtns before we used the constraint in (3).

The constraint in (3) is precisely what we get from applying Lagrange's eqtns to the angular motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) - \frac{\partial L}{\partial \phi} = \boxed{m r^2 \ddot\phi = 0} \qquad (10)$$

The solutions to the eqtns of motion can be written in integral form fairly simply; we now turn to these.

(a) RADIAL DYNAMICS:

One could in principle start from the radial eqtn. of motion in (9), and integrate twice; but it is much simpler to use the conservation of energy in (7), to write

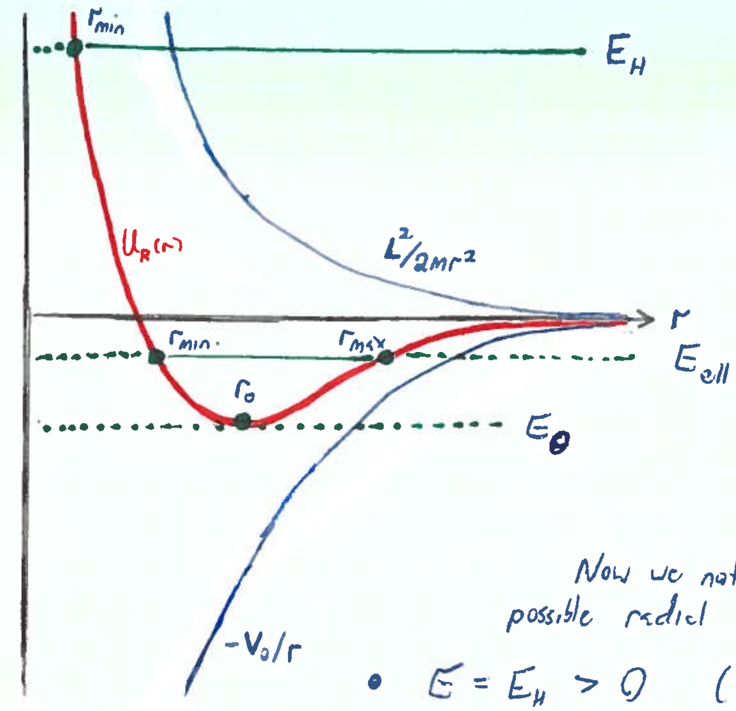
$$\dot r^2 = \frac{2}{m}\left(E - U_R(r)\right) \qquad (11)$$

so that

$$\int dt = \int dr \frac{1}{\sqrt{\frac{2}{m}\left(E - U_R(r)\right)}} \qquad (12)$$

Now integrating this gives us $r(t)$, the radial dynamics. To see how useful eqtn (7) is, let's consider how to use it to see the general characteristics of the motion to be found from (12). What (7) says is that the particle moves in a radial potential $U_R(r)$, and so we should plot this potential to see what happens.

The next page shows us what we get.



NEWTONIAN POTENTIAL

Here we pick a potential

$$V(r) = -V_0/r \qquad (13)$$

with $V_0 > 0$. The effective radial potential, shown in red, is

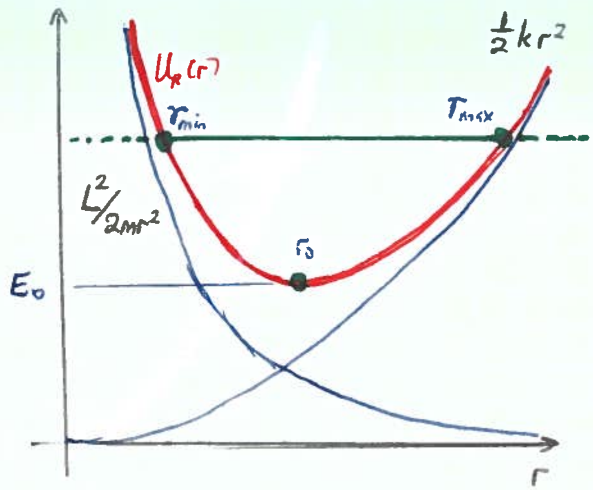
$$U_R(r) = \frac{L^2}{2mr^2} - V_0/r \qquad (14)$$

Now we notice that there are 4 kinds of possible radial motion, viz.

- $E = E_H > 0$ (Hyperbolic motion): Here the motion is unbounded - the radial coordinate $r(t)$ "bounces off" the centrifugal potential at $r = r_{min}$, and otherwise escapes to $r = \infty$, still with positive energy (which, when $r = \infty$, is all kinetic energy).
- $E = 0$ (parabolic motion): This is the same as hyperbolic motion, except that since $E = 0$, the kinetic energy $\to 0$ as $r \to \infty$.
- $0 > E > E_0$ (elliptic motion): Here the particle is bound, such that $r_{max} > r > r_{min}$. The radial coordinate $r(t)$ will now oscillate back and forth between these two limiting values.
- $E = E_0$ (circular motion): The energy is at the minimum possible value, at which the radial contribution to the kinetic energy is $T_r = 0$. Anything less is impossible. For a given value of angular momentum L, this puts the radial coordinate at a value $r = r_0$. One then has

$$\left.\begin{aligned} r_0 &= L^2/mV_0 \\ E_0 &= U_R(r_0) = -\frac{mV_0^2}{2L^2} \end{aligned}\right\} \qquad (15)$$

As we shall discuss below, one can derive all characteristics of the motion in a Newtonian potential by simple integrations.



HARMONIC POTENTIAL

Here we pick a potential

$$V(r) = \tfrac{1}{2}kr^2 \qquad (16)$$

with $k>0$. Since this is always positive, the effective radial potential, shown in red, is also always positive:

$$U_R(r) = \frac{L^2}{2mr^2} + \tfrac{1}{2}kr^2 \qquad (17)$$

Now there are only 2 kinds of motion in the radial coordinate, and they are both bounded; we have

- $E = E_0$ (circular motion): Here we have r at the minimum of the effective potential, ie., at $r = r_0$, where

$$\left.\begin{aligned} r_0 &= \left(L^2/mk\right)^{1/4} \\ U_R(r_0) &= \tfrac{1}{2}(k/m)^{1/2}L = \tfrac{1}{2}\omega_0 L \end{aligned}\right\} \qquad (18)$$

where $\omega_0 = (k/m)^{1/2}$ is the frequency of oscillations in the radial potential.

- $E > E_0$: Again, the radial coordinate oscillates in the potential well around $r = r_0$, between the minimum and maximum values, ie., $r_{max} > r > r_{min}$.

Although we shall not prove it here, there is an interesting result, sometimes called "Bertrand's theorem", to the effect that it is only for the above 2 potentials (the Newtonian and the harmonic potential) that we get closed orbits, ie., the particle orbit "joins up" with itself. This result is rather important in practise, since it means that any small departures from a closed orbit are coming from weak corrections to one or other of these 2 potentials (which are themselves very common in Nature).

(b) ORBITAL DYNAMICS :

In general one would like to find the full solution to the equations of motion; this means finding $\underline{r}(t)$, given some initial condition. In practise what we do is find first the radial solution $r(t)$, and then using the conservation of angular momentum in the form (3), we find $\phi(t)$; this then also gives us the "shape" $r(\phi)$ of the orbit. Collecting the results, we have, from (12), that

$$t - t_0 = \int_{r_0}^{r} dx \, \frac{1}{\sqrt{\frac{2}{m}(E - U_R(x))}} \tag{19}$$

and, from (3), which gives $d\phi = dt/mr^2$, we have, using (12), that

$$\phi - \phi_0 = \int_{r_0}^{r} \frac{dx}{x^2} \frac{L}{\sqrt{2m(E - U_R(x))}} \tag{20}$$

where we have initial conditions $r = r_0$ and $\phi = \phi_0$ (NB: do not confuse r_0 given here with the radius r_0 of circular motion given in (15) and in (18)).

In principle we can extract everything we want from (19) and (20) provided we are able to do the integrals. Rather than give a general analysis, we look at one really important case.

(c) MOTION IN A NEWTONIAN POTENTIAL

: Since the book actually covers the problem of Newtonian dynamics pretty thoroughly, I will simply summarize here the main points.

- The equation (20) for the shape, written in differential form is

$$d\phi = \frac{dr}{r^2} \frac{L}{\sqrt{2m(E + V_0/r - L^2/2mr^2)}} \tag{21}$$

Noting that with the substitution $u = 1/r$, we have $dr/r^2 = du$, we can write this as

$$d\phi = \frac{-du}{\sqrt{2m(E + V_0 u - L^2/2m \, u^2)}} \tag{22}$$

and simply integrate this up; or, in a pretty much equivalent move, we use the same substitution in the radial eqtn in (9), to get the differential eqtn

$$\frac{d^2u}{d\phi^2} + u = V_0 m/L^2 \tag{23}$$

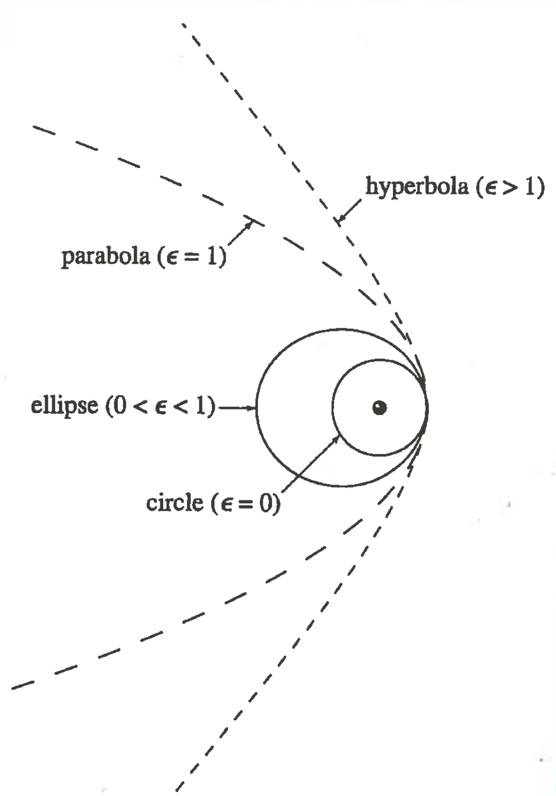
(this is what is done in the book); and of course there are many other ways to integrate the eqtns. of motion in a central force system*

In any case, the solution to either (22) or (23) is

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)} \tag{24}$$

where ϕ_0 is an angle which defines the major axis of symmetry of the system, where r_0 was given in (15), and where e is the eccentricity.

In what follows I summarize the key properties of the solutions for different values of the eccentricity e.



Formally the eccentricity is given by the explicit solution of (22) or (23) to be

$$e = \sqrt{\left(1 + 2EL^2/mV_0^2\right)} \tag{25}$$

and as noted above, in discussing the solutions to the radial equation, they take different forms depending on whether $E > 0$ (hyperbola, with $e > 1$), $E = 0$ (parabola, with $e = 1$), or $E < 0$ (either ellipse, with $1 > e > 0$), or circle, when $E = -mV_0^2/2L^2$, as noted in eqtn (15).

At left we see a collection of such orbits all having the same r_{min}; obviously they all have different energies. Their properties are as follows:

- <u>ELLIPTICAL ORBIT</u> : We can parametrize the elliptical orbit using either an algebraic equation in Cartesian coordinates, or trigonometrically, using eqtn (24) above. In Cartesian coordinates we have (PTO):

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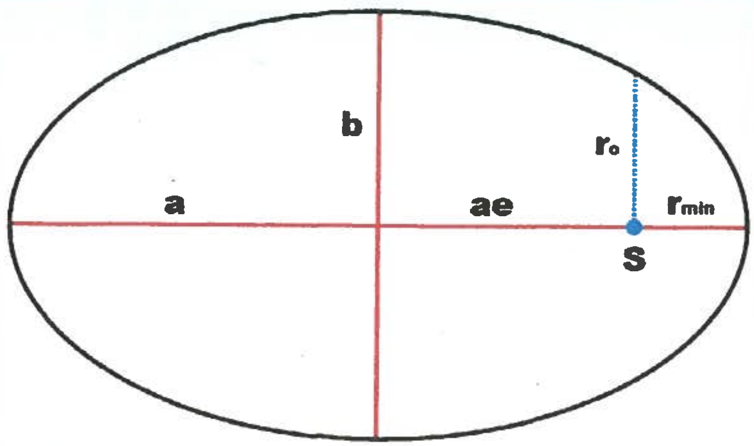
* There is even a book, which discusses the many approaches that have been used to look this problem over the centuries: see P. Colwell, "Solving Kepler's Equation over 3 centuries" (Willmann-Bell, 1993).



$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (26)$$

where a is the semi-major axis, and b the semi-minor axis, as shown at left. In terms of the original parameters of the theory, we have

$$a = \frac{r_0}{1-e^2} = \frac{V_0}{2|E|} \qquad (27)$$

and also

$$b = \sqrt{a r_0} = \frac{r_0}{\sqrt{1-e^2}} = \frac{L}{\sqrt{2m|E|}} \qquad (28)$$

so that b is the geometric mean of a and r_0 (with r_0 the smallest). In terms of the same parameters, we also have

$$\left.\begin{aligned} r_{min} &= \frac{r_0}{1+e} = a(1-e) \\ r_{max} &= \frac{r_0}{1-e} = a(1+e) \end{aligned}\right\} \qquad (29)$$

Clearly, when $e=0$, we get circular motion, at velocity $v = \left(\frac{V_0}{m r_0}\right)^{1/2} = \frac{2\pi r_0}{T}$ (30) where T is the orbital period.

- <u>PARABOLIC ORBIT</u> : When $e=1$, the general eqtn. (24) for the orbits has a divergence when $\phi \to \phi_0$; ie., $r \to \infty$ as $\phi \to \phi_0$. Formally, we see from (29) that $a \to \infty$, $r_{max} \to \infty$, and $r_{min} = r_0/2$. In Cartesian coordinates one finds that

$$y^2 = r_0^2 - 2r_0 x \qquad (31)$$

Note that the velocity V_{max} at perigee, when $r = r_0/2$, is given by

$$V_{max} = 2\left(V_0/m r_0\right)^{1/2} \qquad (32)$$

which $\sqrt{2}$ larger than the orbital speed of an object moving in a circular orbit at the same radius $r_0/2$ from the central potential source.

• <u>HYPERBOLIC ORBIT</u> : Now, with $e > 1$, the eqtn. defining the radius r as a function of ϕ, in (24), has no solution when $e\cos(\phi-\phi_0) \geq 1$, ie., there is a range of angles for which

$$\cos(\phi-\phi_0) \geq 1/e . \qquad (33)$$

where the system cannot go — that this must be the case is obvious from the figure showing the different possible orbits. The hyperbola can be written in Cartesian coordinates as the solution of the eqtn.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (34)$$

where now the eccentricity $e = (1 + b^2/a^2)^{1/2}$. The distance of closest approach by the orbiting system is then $r_{min} = r_0/(1+e)$

One can go into great detail in the analysis of these orbits - this is done in specialized texts. The material is of course mostly very old - these matters were worked out centuries ago.

We note that Kepler's 3 laws are easily justified using the results above. The 1st law is just the idea that closed orbits are ellipses. The 2nd law states that equal areas, measured from the central focus, are swept out in equal times. But this just a statement of angular momentum conservation; the rate at which area is swept out is $dA/dt = \dot{A} = \frac{r^2}{2} d\phi/dt$, so from (3) we have:

$$\dot{A} = L/2m \qquad \text{(Kepler's 2nd law)} \qquad (35)$$

The 3rd law relates the orbital period to the orbital size, and says that the period $T \propto r_0^{3/2}$. To show this for the elliptic orbit, we integrate (35) over a single period, viz.

$$T = \int_0^T dt = \frac{2m}{L}\int dA = \frac{2m}{L}(\pi ab) \qquad (36)$$

and then, using $r_0 = L^2/mV_0$, so $L^2 = mV_0 r_0$, we square (36) to get

$$\left.\begin{aligned} T^2 = \frac{4\pi^2 m (ab)^2}{r_0 V_0} &= 4\pi^2 \frac{m}{V_0} r_0^3/(1-e^2)^{3/2} = 4\pi^2 \frac{m}{V_0} a^3 \\ &= \pi^2 V_0^2 \frac{m}{2|E|^3} \end{aligned}\right\} (37)$$

This concludes our discussion.