DAMPED & DRIVEN OSCILLATORS

To treat damped oscillators by Lagrangian methods is difficult - this is because the Lagrangian method describes CLOSED SYSTEMS, and the friction in damped oscillators is coming from interaction of the coordinates of interest with a large number of other "environmental" degrees of freedom. So problems of damped systems are often dealt with by writing "phenomenological" eqtns of motion, in which friction terms are inserted by hand, often using experiments as a justification for the forms chosen.

In what follows I will (i) give a brief discussion of the physics of friction, showing how this can lead to a Lagrangian description (which, although complicated, is very illuminating); then (ii) I will discuss how one solves the dissipative eqtns of motion for both a single damped oscillator, and then for a set of coupled damped oscillators; finally (iii) I discuss what happens when we add an external driving force to the damped oscillator (or set of oscillators).

The material is mostly covered in Chapter 5 of the text by Taylor.

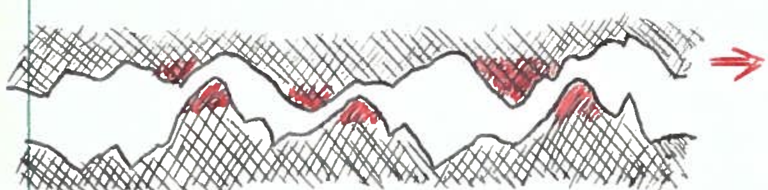
THE PHYSICS of FRICTION :

Whenever we try to move an object through a fluid or a gas, or move one solid surface against another, we always encounter friction. How can this be explained? Let's note several relevant characteristics of the frictions.

(i) In a fluid or gas, the dissipation of energy is clearly accompanied by an irreversible distortion or change of the fluid/gas motion, in the form of waves and/or turbulent motion; think of the wake of a ship, where waves are generated and radiate irreversibly away, and turbulent motion (eddies, vortices) are created which eventually decay into microscopic random fluid/gas motion, which we eventually perceive as heat.

(ii) When we move one solid across another, sound is produced, and this is both inside the solids and in the air; heat is also produced. Thus, as for fluids & gases, we see energy dissipated in the form of both macroscopic wave motion, and microscopic random motion.

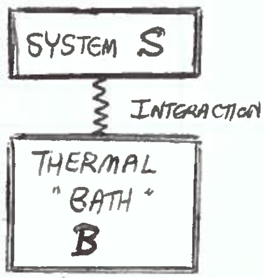
It is common to argue that "solid on solid" friction has to do with the non-uniform shape of the surfaces. It is clear that if we try to move the top surface, shown at left, to the right, then it will "snag" on the bottom surface - the danger points are shown in red. However, notice that if the surfaces move past each other and simply collide, with no distortion of each surface, then no energy will be dissipated - the final state of each surface afterwards will be



the same as their initial states, and so no energy will have been lost to any other degrees of freedom. What actually happens, of course, is that the projections or "outcrops" shown in red are strongly deformed by the snags/collisions, and vibrate because of this — and these vibrations are then transmitted into the bulk solid, and even into the air around.

From what we have seen of the way in which balls and springs work when coupled together – they develop normal mode excitations which look like independent oscillators – we see that an approximate representation of a classical "environment" could be guessed to look like a set of oscillators, representing the normal modes of a solid, liquid, or gas. This then suggests the following kind of Lagrangian to describe friction:

$$\mathcal{L} = L_S(\underline{q}, \dot{\underline{q}}) + L_{int}(\underline{q}, \dot{\underline{q}}; \{x_k, \dot{x}_k\}) + L_B(\{x_k, \dot{x}_k\}) \qquad (1)$$



with interactions described by L_{int} between coordinates $\underline{q}$ describing the degrees of freedom of interest (eg., the centre of mass of some moving solid), and the oscillator coordinates $\{x_k\} = (x_1, x_2, \ldots x_N)$ describing the "thermal bath", ie., all the environmental coordinates that take up the frictional energy. One such model could have

$$\left.\begin{aligned} L_S(\underline{q}, \dot{\underline{q}}) &= \tfrac{1}{2} M_0 \dot{\underline{q}}^2 - U(\underline{q}) \\ L_{int}(\underline{q}, \dot{\underline{q}}; \{x_k, \dot{x}_k\}) &= -\sum_k \underline{V}_k \dot{\underline{q}}\, x_k \\ L_B(\{x_k, \dot{x}_k\}) &= \tfrac{1}{2}\sum_k m_k(\dot{x}_k^2 - \omega_k^2 x_k^2) \end{aligned}\right\} \qquad (2)$$

Now in this model we actually couple the system velocity $\dot{\underline{q}}$ to the oscillator coordinates – notice we have a bilinear coupling, of exactly the kind that we discussed when discussing coupled oscillators.

It is an interesting fact that for SLOW motions of the system, when $\dot{\underline{q}}$ is sufficiently small, this Lagrangian leads to the following equation of motion for $q(t)$:

$$M_0 \ddot{q} + \eta \dot{q} + \partial V/\partial q = F(t) \qquad (3)$$

where $F(t)$ is some external force acting on $q(t)$, and η is the friction coefficient – in electronic circuit theory it would be called the "Ohmic resistance" (with $\dot{q}$ now the current, and q the charge).

Let us now consider how to solve equations like (3); we will focus on the case where $V(q)$ describes an oscillator.

DAMPED HARMONIC OSCILLATOR:

Since you have all looked at the damped oscillator in previous courses, this will be quick. We assume an eqtn. of motion.

$$\left.\begin{aligned} & m\ddot{q} + \eta\dot{q} + kq = 0. \\ \text{or} \quad & \ddot{q} + 2\gamma\dot{q} + \omega_o^2 q = 0 \qquad (2\gamma = \eta/m\ ;\ \omega_o^2 = k/m) \end{aligned}\right\} \quad (4)$$

If we make the ansatz $\quad q \propto e^{st} \quad$ (5)

then $\quad (s^2 + 2\gamma s + \omega_o^2) = 0. \quad$ (6)

so that the values of s are

$$s_{\pm} = \begin{cases} -\gamma \pm (\gamma^2 - \omega_o^2)^{1/2} & \gamma > \omega_o \\ -\gamma \pm i(\omega_o^2 - \gamma^2)^{1/2} & \gamma < \omega_o \end{cases} \quad (7)$$

where $\gamma > \omega_o$ is overdamped, and $\gamma < \omega_o$ is underdamped; when $\gamma = \omega_o$, we have critical damping. The solutions are then.

(i) UNDERDAMPED $(\gamma < \omega_o)$: We have $s_{\pm} = -\gamma \pm i(\omega_o^2 - \gamma^2)$, ie., complex roots; we have a general sol^n to this homogeneous problem going like $q(t) = A_+ e^{s_+ t} + A_- e^{s_- t}$, which we rewrite as

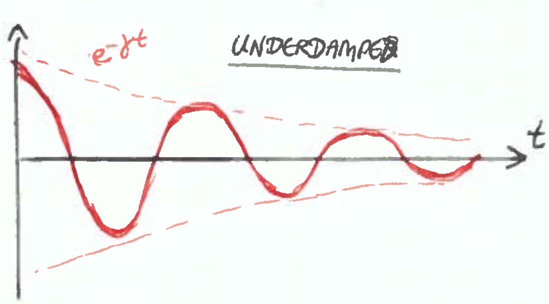
$$q(t) = A e^{-\gamma t} \cos(\Omega_o t + \phi) \qquad (\Omega_o = (\omega_o^2 - \gamma^2)^{1/2}) \quad (8)$$

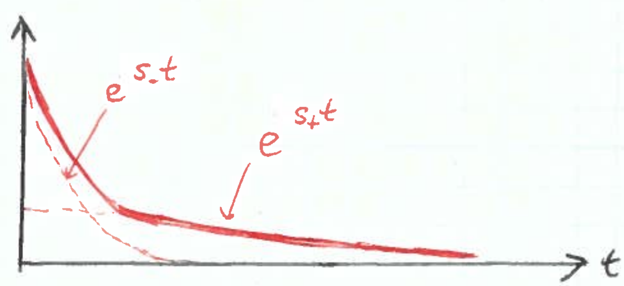
where A and ϕ are arbitrary constants, to be fixed using the 2 boundary conditions (eg., the values of $q(t=0)$ and $\dot{q}(t=0)$). Note that $\Omega_o < \omega_o$; the frequency is always reduced by the damping.

(ii) OVERDAMPED $(\gamma > \omega_o)$: Since now the 2 roots $s_{\pm} = -\gamma \pm (\gamma^2 - \omega_o^2)^{1/2}$ are both real, we have the general solution We then have the general sol^n:

$$q(t) = e^{-\gamma t}\left[A_+ e^{\Omega_o t} + A_- e^{-\Omega_o t}\right] \quad (9)$$

These 2 sol^{ns} look like:





Finally we have the critically damped case, viz

(iii) CRITICALLY DAMPED ($\gamma = \omega_0$) : The 2 roots $s_\pm$ then become degenerate, ie., $s_\pm \to \gamma$. The solution is

$$q(t) = (A_1 + A_2 t)\, e^{-\gamma t} \tag{10}$$

This summarizes results for the damped simple oscillator.

COUPLED DAMPED OSCILLATORS :

This case is a simple generalization of the problem of a set of undamped oscillators. Recall that for a set of N undamped oscillators, we had the coupled eqtns of motion

$$\sum_\beta \left(t_{\alpha\beta}\, \ddot{\underline{q}}_\beta + k_{\alpha\beta}\, \underline{q}_\beta \right) = 0 \qquad (\forall \alpha = 1, 2, \dots N) \tag{11}$$

which in the case where the kinetic energy term in the Lagrangian was of the form $\frac{1}{2}\sum_{\alpha\beta} t_{\alpha\beta}\, \dot{q}_\alpha \dot{q}_\beta$. In the case where the kinetic term comes from simple masses, this kinetic term is just $T = \frac{1}{2}\sum_\alpha m_\alpha \dot{q}_\alpha^2$, and so the above eqtns. become

$$m_\alpha \ddot{\underline{q}}_\alpha + \sum_{\beta=1}^{N} k_{\alpha\beta}\, \underline{q}_\beta = 0 \qquad (\forall \alpha) \tag{12}$$

When we add dissipation, this means adding a frictional force acting on the α-th oscillator; in its most general form this can involve the other oscillators, and this will look like a force linear in $\dot{q}_\alpha$, of form:

$$\underline{f}_\alpha = -\sum_\beta \eta_{\alpha\beta}\, \dot{\underline{q}}_\beta \tag{13}$$

but if we simply assume that we are dealing with a set of masses m_α coupled by springs, then the friction will act individually on different masses, so that

$$\underline{f}_\alpha \to -\eta_\alpha\, \dot{\underline{q}}_\alpha \tag{14}$$

Thus we now end up with a set of coupled equations of form

$$\sum_\beta \left(t_{\alpha\beta}\, \ddot{\underline{q}}_\beta + \eta_{\alpha\beta}\, \dot{\underline{q}}_\beta + k_{\alpha\beta}\, \underline{q}_\beta \right) = 0 \qquad \text{(General case)} \tag{15}$$

$$m_\alpha \ddot{\underline{q}}_\alpha + \eta_\alpha \dot{\underline{q}}_\alpha + \sum_\beta k_{\alpha\beta}\, \underline{q}_\beta = 0 \qquad \text{(masses coupled by springs)} \tag{16}$$

The simplest example of such a problem would be the pair of coupled oscillators we looked at extensively before; in this case we would have a pair of coupled equations of form

$$\left.\begin{aligned} m_1 \ddot{q}_1 + \eta_1 \dot{q}_1 + k_1 q_1 + k_{12} q_2 &= 0 \\ m_2 \ddot{q}_2 + \eta_2 \dot{q}_2 + k_2 q_2 + k_{21} q_1 &= 0 \end{aligned}\right\} \tag{17}$$

where $k_{12} = k_{21}$ is the spring constant for the spring coupling the 2 oscillators, and k_1 and k_2 are the spring constants for the individual springs for each oscillator.

The solution of equations like (15) and (16) uses exactly the same method as for non-dissipative coupled oscillators (cf eqtns. (32)-(42) in the section on coupled oscillators). Thus, we make the substitution

$$Q(t) = \begin{pmatrix} q_1(t) \\ \vdots \\ q_N(t) \end{pmatrix} \equiv q_\alpha(t) = q_\alpha e^{st} \tag{18}$$

instead of (32), and thereby get

$$\left.\begin{aligned} \sum_\beta (t_{\alpha\beta} s^2 + \eta_{\alpha\beta} s + k_{\alpha\beta}) X_\beta &= 0 \qquad (\text{general}) \\ \sum_\beta [(m_\alpha s^2 + \eta_\alpha s)\delta_{\alpha\beta} + k_{\alpha\beta}] X_\beta &= 0 \qquad \left(\begin{matrix}\text{coupled}\\ \text{masses}\end{matrix}\right) \end{aligned}\right\} \text{for each } \alpha \tag{19}$$

with eigenvalues given by

$$\left.\begin{aligned} \det |t_{\alpha\beta} s^2 + \eta_{\alpha\beta} s + k_{\alpha\beta}| &= 0 \qquad (\text{general}) \\ \det |(m_\alpha s^2 + \eta_\alpha s)\delta_{\alpha\beta} + k_{\alpha\beta}| &= 0 \qquad \left(\begin{matrix}\text{coupled}\\ \text{masses}\end{matrix}\right) \end{aligned}\right\} \tag{20}$$

and with solutions

$$\left.\begin{aligned} q_\alpha &= \mathrm{Re} \sum_\beta \Delta_{\alpha\beta} C_\beta e^{s_\beta t} \\ &\equiv \sum_\beta \Delta_{\alpha\beta} Z_\beta(t) \end{aligned}\right\} \tag{21}$$

where now the eigenfunctions $Z_\beta(t)$ satisfy the eqtns

$$\ddot{Z}_\beta + 2\gamma_\beta \dot{Z}_\beta + \omega_\beta^2 Z_\beta = 0 \qquad (\text{for each } \beta) \tag{22}$$

and we write

$$Z_\beta(t) = \left(a_\beta^+ e^{s_\beta^+ t} + a_\beta^- e^{s_\beta^- t}\right) \qquad (\text{for each } \beta) \tag{23}$$

where the coefficients $a_\beta^\pm$ are real, and where the "frequencies" $s_\beta^\pm$ are either real and negative (decaying solutions) or else complex conjugate pairs - indeed, the solution for each normal mode looks just like that for the single damped oscillator we just studied above.

DRIVEN OSCILLATORS & RESONANCE : If we drive an undamped oscillator with an external force $F(t)$, then its oscillation amplitude will typically grow without limit if the force continues to be applied. This is physically unrealistic – damping will always limit this growth. If $F(t)$ is PERIODIC, then the phenomenon of resonance can occur – if the driving frequency matches the natural frequency of the oscillator.

We study this by looking at a single damped oscillator. Thus we look at an equation of motion

$$\left.\begin{aligned} M\ddot{q} + \eta\dot{q} + kq &= F(t) \\ \text{or} \quad \ddot{q} + 2\gamma\dot{q} + \omega_0^2 q &= \bar{f}(t) \end{aligned}\right\} \quad \begin{aligned} 2\gamma &= \eta/m \\ \bar{f}(t) &= F(t)/m \end{aligned} \tag{24}$$

We wish to solve this eqtn for some arbitrary $\bar{f}(t)$, in general; but we will concentrate on the problem when $\bar{f}(t)$ is periodic. We will look at the problem using 2 different methods

Method 1 : The standard way of solving a linear inhomogeneous differential eqtn is to (i) find a "particular integral" of the inhomogeneous eqtn, and then (ii) add to this the general solution of the homogeneous eqtn. Thus we say that for some differential eqtn of form

$$\hat{D}\,q(t) = \bar{f}(t) \tag{25}$$

the solution is

$$\left. q(t) = \underbrace{p(t)}_{\text{particular integral}} + \alpha\,\underbrace{x(t)}_{\text{complementary function}} \right\} \tag{26}$$

where

$$\left.\begin{aligned} \hat{D}p(t) &= \bar{f}(t) \\ \hat{D}x(t) &= 0 \end{aligned}\right\} \tag{27}$$

The general idea is that $\alpha x(t)$ can be added to $p(t)$, and the result will still satisfy (25); and we can generate all solutions to (25) in this way.

Now we already know the complementary function $x(t)$ for (24); it was given in detail for different values of γ/ω_0 in eqtns. (8)-(10). So, for some given form for $\bar{f}(t)$ in (24), we can find $q(t)$ provided we can find just one particular integral.

<u>EXAMPLE</u> : Suppose we drive the system with a periodic force - let us use

$$\bar{f}(t) = \bar{f}_0 \cos\Omega t \tag{28}$$

where Ω is the frequency of the driving force. We then have the problem of finding a particular integral for this problem. Let us write an ansatz for this particular integral in the form

$$x(t) = A e^{i\Omega t} \tag{29}$$

recognizing that it ought to oscillate at the same frequency as the driving force. Simple substitution of (29) into (24) and (28) gives

$$A_0 = \frac{\bar{f}_0}{\omega_0^2 - \Omega^2 + 2i\gamma\omega} \tag{30}$$

which we can rewrite as

$$\left.\begin{aligned} A_0 &= a_0 e^{i\theta_0} \\ \text{with} \quad a_0 &= \frac{\bar{f}_0}{\left[(\omega_0^2-\Omega^2)^2 + 4\gamma^2\Omega^2\right]^{1/2}} \\ \theta_0 &= \tan^{-1}\frac{2\gamma\Omega}{\Omega^2-\omega_0^2} \end{aligned}\right\} \tag{31}$$

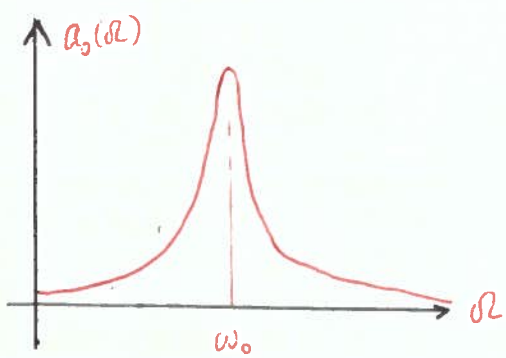
If we now add the complementary solution to this, we get the general solution. Let's do this for the physically interesting case where $\gamma < \omega_0$ (underdamped). Then we have

$$q(t) = A_0 \cos(\Omega_0 t + \theta_0) + A_1 e^{-\lambda t}\cos(\Omega t + \theta_1) \tag{32}$$

where we have taken the real part of $e^{i\theta_0}$ in the 1st term, and added the complementary integral from (8); again, from (8), we have

$$\Omega_0^2 = \omega_0^2 - \gamma^2 \tag{33}$$

We see that there are 2 parts to this solution. The 1st part simply oscillates at the driving frequency; the second part decays, and so eventually goes away, and its oscillatory part oscillates at the slower frequency Ω_0. This latter part is a TRANSIENT: it dies away, and its specific form depends on the initial conditions.



The resonant form for $a_0(\Omega)$ is obvious from the graph.

We see that the amplitude $A_0(\Omega)$ of the long-term oscillations is a maximum at $\Omega = \omega_0$; we have

$$A_0^{max} \sim f_0/2\gamma\omega_0 \tag{34}$$

On the other hand the WIDTH is 2γ; and the "Q-factor" is given by

$$Q = \omega_0/2\gamma \tag{35}$$