COUPLED OSCILLATORS

There is an entire chapter in the book by Taylor which covers this topic. The treatment in the book is overly lengthy and based mostly around examples. Sections 11.5 and 11.7 are quite useful, and you may find the coverage of weakly coupled oscillators (section 11.3) and the double pendulum (section 11.4) quite interesting.

The SIMPLE HARMONIC OSCILLATOR: In what follows I will distinguish between a simple harmonic oscillator, or "SHO", which shows harmonic motion, and a more general non-linear oscillator, which does not. An oscillator is something which oscillates, no more, no less.

The SHO has the Lagrangian

$$L = \frac{M}{2}\left(\dot{q}^2 - \omega_0^2 q^2\right) \tag{1}$$

for a 1-d oscillator. The eqtn. of motion is then

$$\ddot{q} + \omega_0^2 q = 0. \tag{2}$$

and the solution is just

$$q(t) = A_0 \cos(\omega_0 t + \phi_0) \tag{3}$$

where A_0 and ϕ_0 are two undetermined constants, which have to be found knowing 2 "boundary conditions", or initial conditions (eg., the position $q(t_0)$ and the velocity $\dot{q}(t_0)$ at some time t_0). We can also write

$$q(t) = A_1 \cos \omega_0 t + A_2 \sin \omega_0 t \tag{4}$$

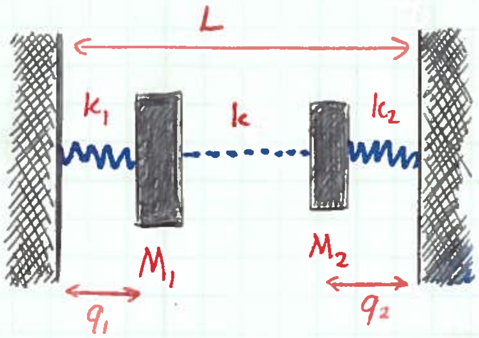
which is equivalent to (3).

One can also have a driven SHO and a damped SHO; we will come to these later.

(a) LAGRANGIANS for COUPLED OSCILLATORS

Oscillators are ubiquitous in Nature - there are many systems in which energy sloshes back and forth between T and V, all the while keeping $E = T + V$ a constant. But oscillators are rarely isolated from each other - there is usually some residual interaction between them. One can see this in a few simple examples involving pairs of oscillators, where there is some physical means by which they couple. Here are a few such examples

1) **HORIZONTAL COUPLED SPRINGS**: Let's consider the situation shown in the figure at left. We have 2 horizontal oscillators, and we assume that the masses M_1 and M_2 and the spring constants k_1 and k_2 are such that we can ignore gravity (if k_1 and k_2 are large enough, this is a good approximation).



In this case, the uncoupled oscillators have the Lagrangian

$$\mathcal{L}^0 = \mathcal{L}_1 + \mathcal{L}_2 = \tfrac{1}{2}\left\{ M_1(\dot{q}_1^2 - \omega_1^2 q_1^2) + M_2(\dot{q}_2^2 - \omega_2^2 q_2^2) \right\} \qquad (5)$$

with eqtns of motion

$$\left.\begin{aligned} \ddot{q}_1 + \omega_1^2 q_1 &= 0 \\ \ddot{q}_2 + \omega_2^2 q_2 &= 0 \end{aligned}\right\} \qquad (6)$$

and where $\omega_1^2 = k_1/M_1$, and $\omega_2^2 = k_2/M_2$.

Now we add a coupling, caused here by a 3rd spring with spring constant k_3. The Lagrangian becomes

$$\left.\begin{aligned} \mathcal{L} = \mathcal{L}^0 + \mathcal{L}_{int}(q_1, q_2) &= \mathcal{L}^0 - \tfrac{1}{2} k_3 [L - (q_1 + q_2)]^2 \\ &= \mathcal{L}^0 - \tfrac{1}{2} k_3 [L^2 + (q_1^2 + q_2^2) - 2L(q_1 + q_2) + 2q_1 q_2] \end{aligned}\right\} \qquad (7)$$

so that

$$\left.\begin{aligned} \mathcal{L} = \tfrac{1}{2}\Big\{ & M_1\left(\dot{q}_1^2 - (\omega_1^2 + k_3/M_1) q_1^2\right) + M_2\left(\dot{q}_2^2 - (\omega_2^2 + k_3/M_2) q_2^2\right) \\ & - k_3 (q_1 q_2 - L(q_1 + q_2)) \Big\} \end{aligned}\right\} \qquad (8)$$

Now this looks a bit messy, but if we write the eqtns of motion we get

$$\left.\begin{aligned} \ddot{q}_1 + \Omega_1^2 q_1 - \frac{k_3}{M_1}(q_2 - L) &= 0 \\ \ddot{q}_2 + \Omega_2^2 q_2 - \frac{k_3}{M_2}(q_1 - L) &= 0 \end{aligned}\right\} \qquad (9)$$

where $\Omega_1^2 = (\omega_1^2 + k_3/M_1)$ and $\Omega_2^2 = (\omega_2^2 + k_3/M_2)$. Thus we have a set of coupled eqtns for $q_1(t)$ and $q_2(t)$, which we can rewrite as

$$\left.\begin{aligned} \ddot{q}_1 + \Omega_1^2 q_1 - \lambda_1 q_2 &= f_1 \\ \ddot{q}_2 + \Omega_2^2 q_2 - \lambda_2 q_1 &= f_2 \end{aligned}\right\} \quad (10)$$

where the constants are

$$\left.\begin{aligned} \lambda_1 &= \frac{k_3}{M_1} & \qquad f_1 &= -\frac{k_3}{M_1} L \\ \lambda_2 &= k_3/M_2 & \qquad f_2 &= -\frac{k_3}{M_2} L \end{aligned}\right\} \quad (11)$$

so that we have a set of 2 oscillators, with RENORMALIZED or shifted frequencies Ω_1 and Ω_2, and a residual coupling between them. There is also a residual force acting on each one of them - this is because the equilibrium position of each has been shifted by the added 3rd spring.

We can make 2 transformations on these eqtns in order to put them into a completely symmetric form, and get rid of the forces on the right-hand side of (10), as follows:

(i) Define new variables

$$\left.\begin{aligned} x_1 &= q_1 - \bar{q}_1 & \qquad \text{where} \quad \bar{q}_1 &= \frac{g_2 a_1 + a_2}{g_1 g_2 - 1} \\ x_2 &= q_2 - \bar{q}_2 & \qquad \bar{q}_2 &= \frac{g_1 a_2 + a_1}{g_1 g_2 - 1} \end{aligned}\right\} \quad (12)$$

and where the constants are

$$\left.\begin{aligned} g_1 &= \frac{k_3}{M_1 \Omega_1^2} & \qquad g_2 &= \frac{k_3}{M_2 \Omega_2^2} \\ a_1 &= k_3 L/M_1 & \qquad a_2 &= k_3 L/M_2 \end{aligned}\right\} \quad (13)$$

Then the eqtns (10) become

$$\left.\begin{aligned} \ddot{x}_1 + \Omega_1^2 x_1 - \lambda_1 x_2 &= 0 \\ \ddot{x}_2 + \Omega_2^2 x_2 - \lambda_2 x_1 &= 0 \end{aligned}\right\} \quad (14)$$

All we have done here is take account of the shift in the equilibrium positions of the 2 masses, caused by the additional spring; $\bar{q}_1$ and $\bar{q}_2$ are the new equilibrium positions.

(ii) To make the equations in (14) completely symmetric, we rescale the variables; let's scale x_2 so that $x_2 \to y_2$, with

$$\left.\begin{aligned} x_2 &= \left(\frac{\lambda_2}{\lambda_1}\right)^{1/2} y_2 \\ x_1 &= y_1 \end{aligned}\right\} \qquad (15)$$

Then the eqtns (14) become, with $\lambda = (\lambda_1\lambda_2)^{1/2}$

$$\left.\begin{aligned} \ddot{y}_1 + \Omega_1^2 y_1 - \lambda y_2 &= 0 \\ \ddot{y}_2 + \Omega_2^2 y_2 - \lambda y_1 &= 0 \end{aligned}\right\} \qquad (16)$$

which are completely symmetric.

I have gone through this whole exercise so you can see how in principle it is possible to take 2 coupled oscillators, in which the coupling is a quadratic form in the q_1 and q_2, ie. of form

$$L_{int} = A_1 q_1^2 + A_2 q_2^2 + B_1 q_1 + B_2 q_2 + C q_1 q_2 + D \qquad (17)$$

and reduce it to one where the coupling is simply of form

$$L_{int} = \lambda q_1 q_2$$

and the frequencies of the new coordinates are shifted. The new Lagrangian leading to the equations of motion in

$$L(y_1 y_2 ; \dot{y}_1, \dot{y}_2) = \frac{A_o}{2}\left[(\dot{y}_1^2 - \Omega_1^2 y_1^2) + (\dot{y}_2^2 - \Omega_2^2 y_2^2) - \lambda y_1 y_2\right] \qquad (18)$$

where A_o is just a constant (given in this case by a function of M_1 and M_2).

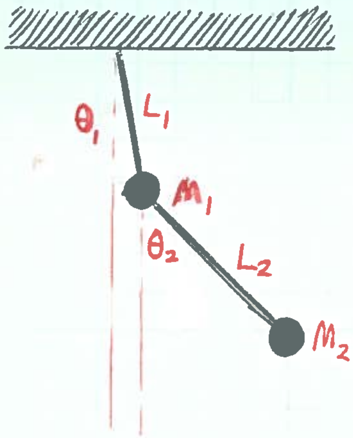
This manoeuvre, of shifting and rescaling the coordinates, can be done for an arbitrarily large number of coupled oscillators, provided the couplings between them are all harmonic, ie., are quadratic forms of the variables. We explore this later on.

Note that we can now easily solve the eqtns of motion in (16); this will also be done shortly. Let us first look at a few other examples

2) DOUBLE PENDULUM: We already looked at this system in our discussion of the form of Lagrangians. From the model answer* to Q. sheet No. 1, we have the total Lagrangian

$$\left.\begin{aligned} L = \tfrac{1}{2}(M_1+M_2)L_1^2\dot{\theta}_1^2 &+ \tfrac{1}{2}M_2 L_2 \dot{\theta}_2^2 + M_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ &+ (M_1+M_2) g L_1 \cos\theta_1 + M_2 g L_2 \cos\theta_2 \end{aligned}\right\} \qquad (19)$$

* See also section 11.4 of the book of Taylor



which is clearly going to lead to very non-linear equations. However we can look at how this problem behaves for small θ_1 and θ_2, and then expand the cosine functions. Thus, assuming that

$$\theta_1, \theta_2 \ll 1 \tag{19}$$

we get a new "small angular displacement" Lagrangian (but note here that in spite of (19), we cannot assume that $\dot\theta_1$ and $\dot\theta_2$ are small); it takes the form

$$\left.\begin{aligned} L(\theta_1,\theta_2;\dot\theta_1,\dot\theta_2) \sim\ & \tfrac{1}{2}(M_1+M_2)L_1^2\dot\theta_1^2 + \tfrac{1}{2}M_2L_2\dot\theta_2^2 \\ & + M_2L_1L_2\dot\theta_1\dot\theta_2\left[1-\tfrac{1}{2}(\theta_1-\theta_2)^2\right] \\ & - \tfrac{1}{2}(M_1+M_2)gL_1\theta_1^2 - M_2gL_2\theta_2^2 \end{aligned}\right\} \tag{20}$$

Notice 2 new features of this Lagrangian. First, we now have a cross interaction between the 2 oscillators (the middle line in (20)) which couples the velocities $\dot\theta_1$ and $\dot\theta_2$ rather than the coordinates θ_1 and θ_2. Second, if we do not neglect the term in $(\theta_1-\theta_2)^2$ in this line, the coupling itself is no longer a quadratic form in θ_1, θ_2, $\dot\theta_2$, but actually a quartic coupling – ie., it is still non-linear. It is common in text discussions of this problem to drop the $(\theta_1-\theta_2)^2$ term, arguing that the product $\dot\theta_1\dot\theta_2$ is small; however strictly speaking this is not true.

From this example we learn that in a coupled oscillator problem we can also have velocity couplings – so we must broaden our consideration of quadratic forms for coupled oscillators to include velocity-dependent potentials, so that we now have a general form

$$\left.\begin{aligned} L(y_1y_2;\dot y_1\dot y_2) = \frac{A_0}{2}\Big[(\dot y_1^2-\Omega_1^2y_1^2) + (\dot y_2^2-\Omega_2 y_2^2) - \lambda_{12}y_1y_2 - \tilde\lambda_{12}\dot y_1\dot y_2 \\ - \tilde\lambda_1\dot y_1y_2 - \lambda_2\dot y_2y_1\Big] \end{aligned}\right\} \tag{21}$$

Actually this form can be simplified, but I will not do this here.

As we will see, an enormous number of real physical systems can be described using Lagrangians for coupled oscillators. This is because for any system having low energy, the leading terms in the expansion of the energy around the minimum are quadratic in the displacement around the minimum. Thus for a 1-dimensional potential we have

$$V(q) = V(q_0) + \tfrac{1}{2} V''(q_0)\,(q-q_0)^2 + \cdots \qquad (22)$$

where $V''(q_0) \equiv d^2V/dq^2|_{q=q_0}$, and $V(q_0)$ is the potential minimum; the linear term in $(q-q_0)$ is zero, because we are at a minimum. Consider now a 2-dimensional oscillator, for which we have the expansion

$$V(\underline{q}) = V(q_1, q_2) = V(\underline{q}_0) + \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \frac{1}{2} \frac{\partial^2 V(\underline{q}_0)}{\partial q_\alpha \, \partial q_\beta} (q^\alpha - q_0^\alpha)(q^\beta - q_0^\beta) + \cdots \qquad (23)$$

which we write as

$$V(\underline{x}) = V(x_1, x_2) = V_0 + \tfrac{1}{2}\left(k_1 x_1^2 + k_2 x_2^2 + 2k_{12}\, x_1 x_2\right) + \cdots \qquad (24)$$

where $x = q - q_0$, with $\alpha = 1, 2$. The Lagrangian for this 2-d oscillator is then just

$$\mathcal{L} = \tfrac{1}{2} M \dot{\underline{x}}^2 - V(\underline{x}) = \tfrac{1}{2} M\left[(\dot{x}_1^2 + \dot{x}_2^2) - (\omega_1^2 x_1^2 + \omega_2^2 x_2^2) - \lambda x_1 x_2\right] \qquad (25)$$

Note that this is hardly different from the Lagrangian for a pair of coupled oscillators, where the coupling is bilinear in x_1 and x_2; the only difference is that for a pair of oscillators, the masses can be different for each oscillator, but this only changes an overall constant in the Lagrangian.

(b) SOLVING THE EQTNS. of MOTION: COUPLED OSCILLATORS

We now wish to solve the eqtns. of motion for a set of coupled oscillators. In what follows I will do the following

(i) Discuss how for Lagrangian in which the potential is expanded up to quadratic order in displacements, we can always "diagonalize" the problem in terms of a set of uncoupled modes, called "normal modes", or "eigenmodes".

(ii) To find these eigenmodes, we find the characteristic eqtn for the coupled eqtns of motion – this is done by a kind of simplified Fourier analysis. From this it is simple to get a complete solution.

In what follows I will first discuss the general theory, and then work out an example of 2 coupled oscillators so that we can see how it works.

Normal Modes for Coupled Oscillators

We begin as above by assuming that the potential energy $V(\underline{\varphi})$ for a set of N oscillators has a minimum at $\underline{\varphi} = \underline{\varphi}_0$, so that we can write

$$V(\underline{\varphi}) = V(\underline{\varphi}_0) + \frac{1}{2}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} \frac{\partial^2 V}{\partial q^\alpha \partial q^\beta}(q^\alpha - q_0^\alpha)(q^\beta - q_0^\beta) + \cdots \qquad (26)$$

which we rewrite as

$$V(\underline{X}) = V_0 + \frac{1}{2}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} k_{\alpha\beta}\, x^\alpha x^\beta + \cdots \qquad (27)$$

where $x^\alpha = q^\alpha - q_0^\alpha$, etc. We also assume that the kinetic energy of the system can be written as

$$T = \frac{1}{2}\sum_{\alpha,\beta=1}^{N}\sum^{N} t_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta \qquad (28)$$

so that

$$L = \frac{1}{2}\sum_{\alpha\beta}\left[t_{\alpha\beta}\, \dot{x}_\alpha \dot{x}_\beta - k_{\alpha\beta}\, x_\alpha x_\beta \right] \qquad (29)$$

The eqtns of motion for this Lagrangian are then given by Lagrange's eqtns., viz

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) - \frac{\partial L}{\partial x_\alpha} = 0 \qquad (30)$$

which gives

$$\sum_{\beta=1}^{N}\left(t_{\alpha\beta}\, \ddot{x}_\beta + k_{\alpha\beta}\, x_\beta \right) = 0 \qquad (\forall \alpha = 1, 2, \ldots N) \qquad (31)$$

ie., we have a set of N coupled linear 2nd-order differential eqtns, which we we wish to solve.

Now the simplest way to solve these eqtns comes from Fourier transform theory, which is discussed elsewhere in these notes. Without going into the details, we can see how it works by making the substitution

$$\underline{X}(t) = \begin{pmatrix} x_1 \\ | \\ x_N \end{pmatrix} \equiv x_\alpha(t) = x_\alpha e^{i\omega t} \qquad (32)$$

where in the end we will take the real part of these quantities. Substituting into (31), and factoring out $e^{i\omega t}$, we get

$$\sum_\beta \left(t_{\alpha\beta}\,\omega^2 - k_{\alpha\beta} \right) x_\beta = 0 \qquad (\text{for each } \alpha) \qquad (33)$$

as a set N coupled algebraic eqtns. For non-vanishing solutions to this set of eqtns we require that

$$\det | \omega^2 t_{\alpha\beta} - k_{\alpha\beta} | = 0 \qquad (34)$$

ie., that

$$\begin{vmatrix} \omega^2 t_{11} - k_{11} & \omega^2 t_{21} - k_{21} & \\ \omega_2^2 t_{12} - k_{12} & \omega^2 t_{22} - k_{22} & \\ & & \ddots \end{vmatrix} = 0 \qquad (35)$$

In the most common case where the kinetic energy term is diagonal, ie., where

$$T = \tfrac{1}{2} \sum_\alpha m_\alpha \dot{x}_\alpha^2 \qquad (36)$$

we have

$$\left.\begin{aligned} \det | \omega^2 m_\alpha \delta_{\alpha\beta} - k_{\alpha\beta} | &= \begin{vmatrix} \omega_2^2 m_1 - k_{11} & -k_{21} & \cdots & -k_{N1} \\ -k_{12} & \omega^2 m_2 - k_{22} & & \vdots \\ \vdots & & \ddots & \\ k_{1N} & & & \omega^2 m_N - k_{NN} \end{vmatrix} \\ &= 0 \end{aligned}\right\} \qquad (37)$$

At this point I am simply going to ASSUME something that I will prove in an appendix, which is that

The solutions to the eqtns (31) can be given in terms of a set of "NORMAL MODES" $z_\alpha(t)$, with associated eigenfrequencies ω_α which are all real positive numbers (ie, $\omega_\alpha \geq 0 \;\; \forall \alpha$), in the form

$$\left.\begin{aligned} x_\alpha(t) &= \mathrm{Re} \sum_\beta \Delta_{\alpha\beta}\, c_\beta\, e^{i\omega_\beta t} \\ &\equiv \sum_\beta \Delta_{\alpha\beta}\, z_\beta(t) \end{aligned}\right\} \qquad (38)$$

and so that the $z_\beta(t)$ satisfy the uncoupled eqtns

$$\ddot{z}_\beta(t) + \omega_\beta^2 z_\beta(t) = 0 \qquad (39)$$

Here the $\Delta_{\alpha\beta}$ are the minors of the determinant in (34), ie, they are the sub-determinants of the matrix $D_{\mu\nu} = (\omega^2 m_{\mu\nu} - k_{\mu\nu})$ obtained by deleting the α-th row and β-th column from $D_{\mu\nu}$, to get a matrix $d^{\alpha\beta}$; then $\Delta_{\alpha\beta} = \det | d^{\alpha\beta} |$.

What this result says is that motion of any one of the masses can be decomposed into a linear superposition of oscillations from the normal modes; each of these normal modes oscillates independently. We are all familiar with these sorts of superposition, for objects ranging from musical instruments to atoms.

Notice that (39) implies we may also write the Lagrangian in the form

$$\mathcal{L} = \frac{1}{2} \sum_\alpha \tilde{m}_\alpha \left(\dot{z}_\alpha^2 - \omega_\alpha^2 z^2 \right) \tag{40}$$

where the $\tilde{m}_\alpha$ are constants have the dimensions of mass. If we write generalized coordinates

$$q_\alpha(t) = \tilde{m}_\alpha^{1/2} z_\alpha(t) \tag{41}$$

then these new coordinates appear in a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2} \sum_\alpha \left(\dot{q}_\alpha^2 - \omega_\alpha^2 q_\alpha^2 \right) \tag{42}$$

Thus we conclude that any bilinear Lagrangian of the form given in (29) can be written as a sum over normal modes as in (42). Actually this conclusion generalizes to any Lagrangian which is a sum of bilinear terms in $\dot{q}_\alpha \dot{q}_\beta$, $q_\alpha q_\beta$, $\dot{q}_\alpha q_\beta$.

EXAMPLE 1 : HORIZONTAL COUPLED SPRINGS : Consider again the example of 2 masses coupled to each other and to stationary walls by horizontal springs, for which we derived the eqtns of motion

$$\left.\begin{aligned} \ddot{y}_1 + \Omega_1^2 y_1 - \lambda y_2 &= 0 \\ \ddot{y}_2 + \Omega_2^2 y_2 - \lambda y_1 &= 0 \end{aligned}\right\} \tag{43}$$

Now this is a special case of (31), with $N = 2$, $m_{\alpha\beta} = \delta_{\alpha\beta}$, and

$$k_{\alpha\alpha} = \Omega_\alpha^2 \qquad k_{12} = k_{21} = -\lambda \tag{44}$$

We now make the substitution given in (32), ie., writing $y_\alpha(t) = Y_\alpha e^{i\omega t}$, we get the algebraic eqtn.

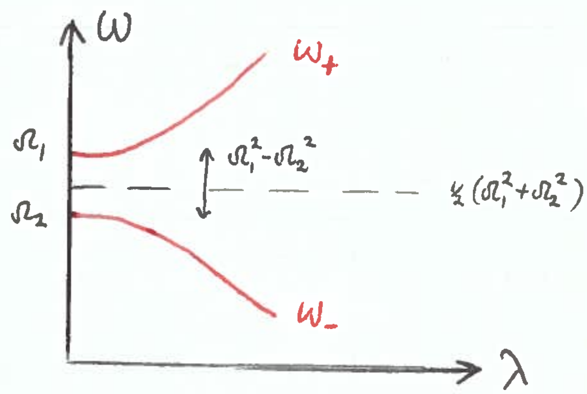
$$\begin{pmatrix} \omega^2 - \Omega_1^2 & \lambda \\ \lambda & \omega_2^2 - \Omega_2^2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = 0 \tag{45}$$

so that we have

$$(\omega^2 - \Omega_1^2)(\omega^2 - \Omega_2^2) - \lambda^2 = 0 \qquad (46)$$

with roots given by

$$\omega_\pm^2 = \tfrac{1}{2}\left\{ (\Omega_1^2 + \Omega_2^2) \pm \left[(\Omega_1^2 - \Omega_2^2) + 4\lambda^2 \right]^{1/2} \right\} \qquad (47)$$



The diagram shows what the eigenvalues do as we switch on the coupling.

Suppose we look at a simpler case of this problem, in which we let

$$\left.\begin{aligned} \Omega_1^2 = \Omega_2^2 = \Omega_0^2 &= \omega_0^2 + k_3/M_3 = \omega_0^2 + \lambda \\ \Omega_0^2 &= k_0/M_0 \end{aligned}\right\} \qquad (48)$$

ie., where we make $M_1 = M_2 = M_0$, so that $k_1 = k_2 = k_0$; and $\omega_0^2 = k_0/M_0$. Then we simply have a coupling k_3 between 2 identical oscillators.
The eqtns then read

$$\begin{pmatrix} \omega^2 - \Omega_0^2 & \lambda \\ \lambda & \omega^2 - \Omega_0^2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0.$$

with roots $\omega_\pm^2 = \Omega_0^2 \pm \lambda$.

$$\Rightarrow \left\{ \begin{aligned} k_0/M_0 &= \omega_- \\ k_0/M_0 + 2k_3/M_3 &= \omega_+ \end{aligned} \right\} \qquad (49)$$

The general solution for the system is then

$$\left.\begin{aligned} y_1(t) &= A_1^+ \cos \omega_+ t + A_1^- \cos \omega_- t \\ y_2(t) &= A_2^+ \cos \omega_+ t + A_2^- \cos \omega_- t \end{aligned}\right\} \qquad (50)$$

where we fix the coefficients $A_\alpha^\pm$ from the initial conditions.