THE FORM of the LAGRANGIAN

The principle of least action gives us Lagrange's eqtns., a set of differential eqtns for $L(\varphi,\dot{\varphi})$ which are the eqtns. of motion of the system. However, these tell us nothing if we don't know what is $L(\varphi,\dot{\varphi})$ for a given system.

So how do we find $L(\varphi,\dot{\varphi})$, for some arbitrary physical system? We start simple, with some symmetry arguments, which leads to a standard form for $L(\varphi,\dot{\varphi})$ under certain restrictions. I then illustrate how this works with some simple examples. Later on we will look at how $L(\varphi,\dot{\varphi})$ takes on somewhat different forms for a more general class of systems.

SYMMETRY ARGUMENTS :

The use of symmetry arguments in physics can often lead to surprisingly specific results, as well as some sweepingly general conclusions. Here are some of them:

(1) Isolated / Closed Systems : We have restricted our attention to closed systems, ie., ones for which no external influences act on the generalized coordinates $(\varphi,\dot{\varphi})$. Under these circumstances we can always define a centre of mass coordinate, which I will call $\underline{X}(t)$. Thus, eg., for a set of particles of masses m_j at positions $\underline{r}_j$, we have

$$M\underline{X} = \sum_j m_j \underline{r}_j \tag{1}$$

where the total mass is

$$M = \sum_j m_j \tag{2}$$

Now, since there are no forces or influences of any kind acting on the total system, it is clear that $L(\underline{X},\dot{\underline{X}})$ cannot depend on $\underline{X}$; the system sees every position the same as every other, and so L is invariant under changes of $\underline{X}$. Thus $L = L(\dot{\underline{X}})$ is a function of $\dot{\underline{X}}$ only.

We can also argue that L cannot depend on the direction of $\dot{\underline{X}}$, since all directions will look the same. The most natural conclusion then is that L depends only on $|\dot{\underline{X}}|^2$, ie., that we can write

$$L(\dot{\underline{X}}) = L(\dot{\underline{X}}^2) = \sum_{n=1}^{\infty} c_n |\dot{\underline{X}}|^{2n} \tag{3}$$

ie., a power series in the velocity squared.*

Experimentally it is found that for velocities $\dot{X} \ll c$, the velocity of light, we have

$$L(\dot{\underline{X}}) = c_2 \dot{X}^2 \longrightarrow \tfrac{1}{2} M \dot{X}^2 \tag{4}$$

* Strictly speaking we could have odd powers of $|\dot{X}|$, but one usually expects to see an analytic function of $\dot{\underline{X}}$ for simplicity's sake.

where the constant C_2 will turn out to be equal to $\frac{1}{2}M_0$. It is found that for higher velocities, there are corrections to this, coming in powers of $(\dot{X}/c)^2$; this is one of the conclusions from the Special Theory of Relativity. In what follows we will stick with (4); this keeps us in the realm of Newtonian classical mechanics.

So – we conclude that for an isolated or "free" system, the Lagrangian is a quadratic function of $\dot{X}$.

<u>Time Invariant Systems</u>: The above argument only applies to the centre of mass — there will still be interactions, in general, between the different internal degrees of freedom of a massive body. Thus, for some system having multiple degrees of freedom (like a multi-particle system) the Lagrangian will still be a function of these coordinates (eg., with interparticle interactions), even though it doesn't depend on the centre of mass coordinate.

However, there are other symmetries that we can invoke. Since $L(\varphi, \dot{\varphi})$ does not depend on t, we know that $\partial L/\partial t = 0$; however it is <u>NOT</u> the case that $dL/dt = 0$, since $\Phi(t)$ and $\dot{\varphi}(t)$ change with time, and

$$\frac{dL}{dt} = \frac{\partial L}{\partial \varphi}\dot{\varphi} + \frac{\partial L}{\partial \dot{\varphi}}\ddot{\varphi} \qquad (5)$$

But there is a quantity that is independent of time. We form the quantity

$$E = \dot{Q}\frac{\partial L}{\partial \dot{\Phi}} - L \qquad (6)$$

We can then easily check that it is independent of time, provided the system is isolated. We have

$$\left.\begin{aligned} \frac{dE}{dt} &= \left[\left(\ddot{\varphi}\frac{\partial L}{\partial \dot{\varphi}} + \dot{\varphi}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right)\right) - \left(\frac{\partial L}{\partial \dot{\varphi}}\ddot{\varphi} + \frac{\partial L}{\partial \varphi}\dot{\varphi}\right)\right] \\ &= \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}}\right) - \frac{\partial L}{\partial \varphi}\right]\dot{Q} \quad = \quad 0 \end{aligned}\right\} \qquad (7)$$

where the last step follows because the quantity in square brackets is just Lagrange's eqtn, viz.:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}}\right) - \frac{\partial L}{\partial \varphi} = 0 \qquad (8)$$

The quantity E is called the energy – we will see that it is equal to the sum of kinetic & potential energies for simple systems.

These results give a idea of what one can accomplish with symmetry arguments. One can derive others like this, but we will come to these later.

INSTANTANEOUS INTERACTIONS : In non-relativistic physics, we deal with objects made up from particles, which move slowly compared to the velocity of light. Under these circumstances it is found, by a combination of experiment and theoretical analysis, that the Lagrangian takes the form

$$L = \tfrac{1}{2}\sum_{j=1}^{P} m_j \dot{\underline{r}}_j^{\,2} - V(\underline{r}_1, \underline{r}_2 \cdots \underline{r}_P) \tag{9}$$

for a set of P particles. The use of the $-$ sign in front of V corresponds to convention – we see from (6) that we must also have

$$E = \tfrac{1}{2}\sum_{j=1}^{P} m_j \dot{\underline{r}}_j^{\,2} + V(\underline{r}_1 \cdots \underline{r}_P) \tag{10}$$

Thus $V(\underline{r}_1 \cdots \underline{r}_P)$ is just the conventional potential energy. We usually write the kinetic energy as T, so that we have

$$\left.\begin{aligned} L &= T - V \\ E &= T + V \end{aligned}\right\} \tag{11}$$

To see how all this works it will be easiest to consider some examples. We notice that Lagranges eqtns applied to L of the form is (9) are given by

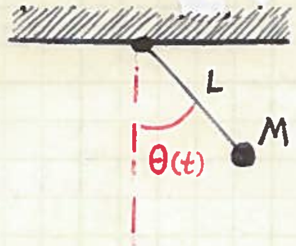
$$m_j \ddot{\underline{r}}_j + \frac{\partial V}{\partial \underline{r}_j} = 0 \tag{12}$$

which is of course just Newton's law $\quad m_j \ddot{\underline{r}}_j = \underline{F}_j \quad (13)$

with $\underline{F}_j = -\partial V / \partial \underline{r}_j$.

Let us now consider some examples.

EXAMPLE ① <u>SIMPLE PENDULUM</u> : This is one of the most basic problem we can look at. The kinetic and potential energies are easily written down; we have the situation shown in the figure, and the Lagrangian is



$$\left. \begin{aligned} L &= T - V \\ &= \tfrac{1}{2} M L^2 \dot{\theta}^2 + M g L \cos\theta \end{aligned} \right\} \qquad (14)$$

where we use $V = -MgL\cos\theta$ (dropping any irrelevant constants).

Note that from this we have $L(\theta, \dot{\theta})$, and the Lagrange eqtn is then just

$$\left. \begin{aligned} & \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} \\ &= M L^2 \ddot{\theta} + M g L \sin\theta \quad = \quad 0 \end{aligned} \right\} \qquad (15)$$

or, simplifying things:

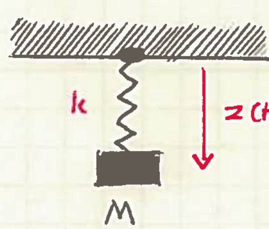
$$\ddot{\theta} + \omega_0^2 \sin\theta(t) = 0 \qquad (16)$$

Notice that if $\sin\theta \ll 1$, then using the usual expansion of $\sin\theta$ as $\sin\theta = \sum_{n=0}^{\infty} (-1)^n \theta^{2n+1}/(2n+1)!$, we have

$$\ddot{\theta} + \omega_0^2 \theta(t) = 0 \qquad (\theta \ll 1) \qquad (17)$$

which is the eqtn of a simple harmonic oscillator (SHO) with frequency ω_0. However, for larger amplitude, the solution is much more complicated, and can be written in terms of elliptic functions.

EXAMPLE 2 <u>SIMPLE OSCILLATOR</u>: This is another classic problem, shown in the figure below left. The mass M is pulled down by gravity, but upwards by a massless spring with spring constant k. Notice that the displacement $z(t)$ is being measured <u>downwards</u>.



The kinetic energy is just

$$T = \tfrac{1}{2} M \dot{z}^2 \qquad (18)$$

and the potential energy is

$$V = \tfrac{1}{2} k z^2 - M g z \qquad (19)$$

(NB: again note that $z(t)$ is measured downwards). Thus the Lagrangian for this system is just $$L = \tfrac{1}{2} M \dot{z}^2 + \left(Mgz - \tfrac{1}{2} k z^2 \right) \qquad (20)$$

Now it makes sense to change variables here — we note that the potential is minimized when

$$z \to z_0 = Mg/k \qquad (21)$$

and so, introducing the variable $$x = z - z_0 = z - Mg/k \qquad (22)$$

we can rewrite the Lagrangian as

$$L = \tfrac{1}{2} M \dot{x}^2 - \tfrac{1}{2} k x^2 \qquad (23)$$

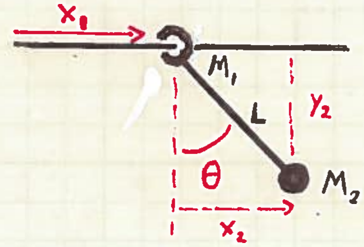
The eqtns. of motion are then what we expect:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = M\ddot{x} + kx = 0 \qquad (24)$$

or more simply:

$$\left.\begin{aligned} \ddot{x} + \omega_0^2 x(t) &= 0 \\ \omega_0^2 &= k/M \end{aligned}\right\} \qquad (25)$$

EXAMPLE 3 **SLIDING PENDULUM**: The previous 2 examples were almost trivial. Let's look now at a problem where we have a pendulum whose support, at position x_1, can actually slide without friction along a wire. Moreover we give the support a mass M_1 and the pendulum itself a mass M_2.



There are several ways to set this problem up. The simplest is to just write the coordinates of the 2 masses as

$$\left.\begin{aligned} M_1 &: \quad (x_1(t),\ 0) \\ M_2 &: \quad (x_1(t) + x_2(t),\ y_2(t)) = (x_1 + L\sin\theta,\ -L\cos\theta) \end{aligned}\right\} \qquad (26)$$

The kinetic energy can then be written as

$$\left.\begin{aligned} T &= \tfrac{1}{2} M_1 \dot{x}_1^2 + \tfrac{1}{2} M_2 \left[(\dot{x}_1 + \dot{x}_2)^2 + \dot{y}_2^2 \right] \\ &= \tfrac{1}{2}(M_1 + M_2)\dot{x}_1^2 + \tfrac{1}{2} M_2 \left[\left(2\dot{x}_1 L \dot{\theta}\cos\theta + L^2\dot{\theta}^2\cos^2\theta \right) + L^2\dot{\theta}^2\sin^2\theta \right] \\ &= \tfrac{1}{2}(M_1 + M_2)\dot{x}_1^2 + \tfrac{1}{2} M_2 \left(L^2\dot{\theta}^2 + 2L\dot{x}_1\dot{\theta}\cos\theta \right) \end{aligned}\right\} \qquad (27)$$

where we use Pythagoras's theorem to separate out the components of $(\text{velocity})^2$, also use $\cos^2\theta + \sin^2\theta = 1$.

Since the potential energy is just: $V = -\frac{1}{2} M_2 g L \cos\theta$ (28)

we have a Lagrangian

$$L = \tfrac{1}{2}(M_1+M_2)\dot{x}_1^2 + \tfrac{1}{2}M_2(L^2\dot{\theta}^2 + 2L\dot{x}_1\dot{\theta}\cos\theta) + \tfrac{1}{2}M_2 gL\cos\theta \qquad (29)$$

We notice 2 things about this Lagrangian. First, getting rid of the mass M_1, by letting $M_1 = 0$, does very little: all the complexity is in the motion of M_2. Second, if we look at Lagrange's eqtns for the variables (ie., for $x_1, \theta, \dot{x}_1, \dot{\theta}$), we have

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1}\right] = (M_1+M_2)\ddot{x}_1 + M_2 L\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = 0. \qquad (30)$$

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}\right] = \frac{d}{dt}\left(M_2 L\left(L\dot{\theta} + \dot{x}_1\cos\theta\right)\right) + M_2 L\left(\dot{x}_1\dot{\theta}\sin\theta + \tfrac{1}{2}g\sin\theta\right)$$

$$= M_2 L\left[L\ddot{\theta} + \ddot{x}_1\cos\theta + \tfrac{1}{2}g\sin\theta\right] = 0 \qquad (31)$$

or, getting rid of unnecessary constants, we have

$$\left.\begin{aligned} \ddot{x}_1 + \frac{M_2 L}{M_1+M_2}\left(\ddot{\theta}\cos\theta + \dot{\theta}^2\sin\theta\right) &= 0 \\ \ddot{\theta} + \tfrac{1}{L}\left(\ddot{x}_1\cos\theta + \tfrac{1}{2}g\sin\theta\right) &= 0 \end{aligned}\right\} \qquad (32)$$

so that we have 2 coupled eqtns for $\theta(t)$ and $x_1(t)$. The first one tells us that the "force" acting on $x_1(t)$ involves a strongly non-linear function of θ, as well as $\dot{\theta}$ and $\ddot{\theta}$ (so that the acceleration of $x_1(t)$ is coupled to the acceleration of $\theta(t)$; and the force on $\theta(t)$ from the 2nd eqtn also involves a non-linear function of $\theta(t)$ (as well as $\ddot{x}_1(t)$).

These eqtns. of motion are not simple to interpret – clearly any motion of $\theta(t)$ will lead to some complex oscillatory force on $x_1(t)$; and any acceleration of $x_1(t)$ will cause $\theta(t)$ to oscillate; and these motions feed back on each other.

There are other ways to get these results, which we will return to once we have discussed motion in rotating frames of reference.