Phys 306 Homework 7 solutions. April 11, 2017

Marking scheme in red.

1 Question 1:

part a)

The Euler equations in this case are:

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0 \tag{1.1}$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = 0 \tag{1.2}$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = 0 \tag{1.3}$$

Now suppose that $I_1 - I_2$ is small we can set up a small dimensionless parameter ϵ as follows ,

$$\epsilon = \frac{I_1 - I_2}{I_3}. \tag{1.4}$$

Note there are other possible choices for the dimensionless parameter which should yield equivalent results (any dimensionless parameter which is proportional to $I_1 - I_2$ will do). To begin with we have 3 parameters I_1, I_2 and I_3 we don't want to add any parameters so we should eliminate one, say I_2 using,

$$I_2 = I_1 - \epsilon I_3. \tag{1.5}$$

The expansions for our angular velocities are as stated in the question

$$\omega_1(t) = \Omega_1(t) + \epsilon\eta_1^{(1)}(t) + \epsilon^2\eta_1^{(2)}(t) + \dots \tag{1.6}$$
$$\omega_2(t) = \Omega_2(t) + \epsilon\eta_2^{(1)}(t) + \epsilon^2\eta_2^{(2)}(t) + \dots \tag{1.7}$$
$$\omega_3(t) = \Omega_3(t) + \epsilon\eta_3^{(1)}(t) + \epsilon^2\eta_3^{(2)}(t) + \dots. \tag{1.8}$$

Plugging equations in (1.5-1.8) into the equations of motion (1.1-1.3) and collecting different powers of ϵ gives

$$I_1\dot{\Omega}_1 + (I_3 - I_1)\Omega_2\Omega_3 + \epsilon\left(I_1\dot{\eta}_1^{(1)} + (I_3 - I_1)\left[\eta_2^{(1)}\Omega_3 + \Omega_2\eta_3^{(1)}\right] + I_3\Omega_2\Omega_3\right) + \epsilon^2(\dots) + \dots = 0 \tag{1.9}$$
$$I_1\dot{\Omega}_2 + (I_1 - I_3)\Omega_3\Omega_1 + \epsilon\left(I_1\dot{\eta}_2^{(1)} + (I_1 - I_3)\left[\eta_3^{(1)}\Omega_1 + \Omega_3\eta_1^{(1)}\right] + I_3\dot{\Omega}_2\right) + \epsilon^2(\dots) + \dots = 0 \tag{1.10}$$
$$I_3\dot{\Omega}_3 + \epsilon I_3\left(\dot{\eta}_2^{(1)} - \Omega_1\Omega_2\right) + \epsilon^2(\dots) + \dots = 0. \tag{1.11}$$

The lowest order approximation is obtained by taking the order ϵ^0 part of the equations (1.9-1.11) (this is equivalent to setting $\epsilon = 0$),

$$\dot{\Omega}_1 - \frac{I_1 - I_3}{I_1}\Omega_2\Omega_3 = 0 \tag{1.12}$$
$$\dot{\Omega}_2 + \frac{I_1 - I_3}{I_1}\Omega_3\Omega_1 = 0 \tag{1.13}$$
$$\dot{\Omega}_3 = 0. \tag{1.14}$$

The lowest order equations are just the equations of motion obtained if $I_2 = I_1$ which is what we expect. Now we have to deal with the initial conditions, the order ϵ^0 part of the initial conditions are (substitute equations (1.6-1.8) into the initial conditions at set $\epsilon = 0$)

$$\Omega_1(0) = \Omega_1^{(0)} \tag{1.15}$$
$$\Omega_2(0) = 0 \tag{1.16}$$
$$\Omega_3(0) = \Omega_3^{(0)} \tag{1.17}$$

the equation for $\Omega_3(t)$ can then be solved,

$$\Omega_3(t) = \Omega_3^{(0)} \text{ constant.} \tag{1.18}$$

putting this solution into (1.12) and (1.13) gives

$$\dot{\Omega}_1 - \frac{I_1 - I_3}{I_1}\Omega_2\Omega_3^{(0)} = 0 \tag{1.19}$$

$$\dot{\Omega}_2 + \frac{I_1 - I_3}{I_1}\Omega_3^{(0)}\Omega_1 = 0. \tag{1.20}$$

Substituting $\Omega_j = e^{i\nu t}a_j$ into the above and solving for the frequency ν we get

$$\nu = \pm\frac{I_1 - I_3}{I_1}\Omega_3^{(0)} = \pm\nu_0. \tag{1.21}$$

So Ω_2 and Ω_1 oscillate at a frequency $\nu_0 = \frac{I_1 - I_3}{I_1}\Omega_3^{(0)}$ which along with the initial conditions (1.15) and (1.16) suggests that the solution to the lowest order solution is (you can verify that these are the solution by differentiating them again and plugging into equations (1.19) and (1.20))

$$\Omega_1(t) = \Omega_1^{(0)}\cos\nu_0 t = \Omega_1^{(0)}\cos\left(\frac{I_1 - I_3}{I_1}\Omega_3^{(0)}t\right) \tag{1.22}$$

$$\Omega_2(t) = -\,\Omega_1^{(0)}\sin\nu_0 t = -\Omega_1^{(0)}\sin\left(\frac{I_1 - I_3}{I_1}\Omega_3^{(0)}t\right). \tag{1.23}$$

Note the minus sign in (1.23). Thus to lowest order the vector $\boldsymbol{\omega}(t)$ is

$$\boxed{\boldsymbol{\omega}(t) \approx \Omega_1^{(0)}\left[\cos\left(\frac{I_1 - I_3}{I_1}\Omega_3^{(0)}t\right)\hat{\mathbf{e}}_1 - \sin\left(\frac{I_1 - I_3}{I_1}\Omega_3^{(0)}t\right)\hat{\mathbf{e}}_2\right] + \Omega_3^{(0)}\hat{\mathbf{e}}_3} \tag{1.24}$$

where $\{\hat{\mathbf{e}}_j\}$ are the unit vectors pointing along the principal axis.

part 2b)

The next order equations of motion (order ϵ^1) are obtained by setting the coefficients of ϵ in the equations (1.9-1.11) equal to zero which gives

$$I_1\dot{\eta}_1^{(1)} + (I_3 - I_1)\left[\eta_2^{(1)}\Omega_3 + \Omega_2\eta_3^{(1)}\right] + I_3\Omega_2\Omega_3 = 0 \tag{1.25}$$

$$I_1\dot{\eta}_2^{(1)} + (I_1 - I_3)\left[\eta_3^{(1)}\Omega_1 + \Omega_3\eta_1^{(1)}\right] + I_3\dot{\Omega}_2 = 0 \tag{1.26}$$

$$\dot{\eta}_2^{(1)} - \Omega_1\Omega_2 = 0. \tag{1.27}$$

Plugging the solutions for Ω_1 (1.22) and Ω_2 (1.23) into (1.27) gives

$$\dot{\eta}_2^{(1)}(t) = -{\Omega_1^{(0)}}^2\cos\nu_0 t\sin\nu_0 t = -\frac{{\Omega_1^{(0)}}^2}{2}\sin 2\nu_0 t \tag{1.28}$$

$$\Rightarrow\ \eta_2^{(1)}(t) = C - \frac{{\Omega_1^{(0)}}^2}{4\nu_0}\cos 2\nu_0 t \tag{1.29}$$

where C is an integration to be determined by the initial condition. Initially $\omega_3(0) = \Omega_3(0) + \epsilon\eta_3^{(1)}(0) + \ldots = \Omega_3^{(0)}$ but $\Omega_3(0) = \Omega_3^{(0)}$ (from equation (1.17)) so $\epsilon\eta_3^{(1)}(0) + \ldots = 0$ which is easily satisfied by putting $\eta_3^{(n)}(0) = 0$ and therefore

$$C - \frac{{\Omega_1^{(0)}}^2}{4\nu_0} = 0 \quad\Rightarrow\quad C = \frac{{\Omega_1^{(0)}}^2}{4\nu_0} \tag{1.30}$$

so

$$\eta_3^{(1)}(t) = \frac{{\Omega_1^{(0)}}^2}{4\nu_0}(1 - \cos 2\nu_0 t) = \frac{{\Omega_1^{(0)}}^2 I_1}{4(I_1 - I_3)\Omega_3^{(0)}}\left[1 - \cos\left(2\frac{I_1 - I_3}{I_1}\Omega_3^{(0)}t\right)\right] \tag{1.31}$$

where I have used the expression for ν_0. Thus the solution for $\omega_3(t)$ to order ϵ is

$$\boxed{\omega_3(t) \approx \Omega_3 + \epsilon\eta_3^{(1)}(t) = \Omega_3^{(0)} + \frac{(I_1 - I_2)I_1\Omega_1^{(0)^2}}{4I_3(I_1 - I_3)\Omega_3^{(0)}}\left(1 - \cos\left(2\frac{I_1 - I_3}{I_1}\Omega_3^{(0)}t\right)\right).} \tag{1.32}$$

(I have used the definition for ϵ in the above).

part 2C)

Cf. section 10.8 Taylor - Classical Mechanics

(i)

In the body frame rotates with the principal axes we can get $\mathbf{L}$ using the formula $L_i = I_{ij}\omega_j$ with a diagonal inertial tensor

$$\mathbf{I}_{\text{body}} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{1.33}$$

So that using the lowest order solution (1.24) for $\boldsymbol{\omega}_{\text{body}}(t)$ we have (remember to lowest order $I_1 = I_2$),

$$\mathbf{L}_{\text{body}} = \mathbf{I}_{\text{body}} \cdot \boldsymbol{\omega}_{\text{body}}(t) = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \Omega_1^{(0)} \cos\nu_0 t \\ -\Omega_1^{(0)} \sin\nu_0 t \\ \Omega_3^{(0)} \end{pmatrix} = \begin{pmatrix} \Omega_1^{(0)} I_1 \cos\nu_0 t \\ -\Omega_1^{(0)} I_1 \sin\nu_0 t \\ I_3\Omega_3^{(0)} \end{pmatrix}. \tag{1.34}$$

Note that $\mathbf{L}$ is time dependent in the body frame even though there are no external torques this is because the body frame is non-inertial. To try and get a better understanding of this we write $\mathbf{L}$ in terms of the unit vectors $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ which point along the principle axes of the body,

$$\boxed{\mathbf{L}(t) \approx \Omega_1^{(0)} I_1 \left[\cos\left(\frac{I_1 - I_3}{I_1}\Omega_3^{(0)}t\right)\hat{\mathbf{e}}_1 - \sin\left(\frac{I_1 - I_3}{I_1}\Omega_3^{(0)}t\right)\hat{\mathbf{e}}_2\right] + \Omega_3^{(0)} I_3 \hat{\mathbf{e}}_3}. \tag{1.35}$$

Now $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ move with the body (and thus depend on time) which is why the components of $\mathbf{L}$ are time dependent even though we expect $\mathbf{L}$ to be conserved. The magnitude of $\mathbf{L}$ in the body frame is constant,

$$|\mathbf{L}_{\text{body}}| = \sqrt{(I_1\Omega_1^{(0)})^2 + (I_3\Omega_3^{(0)})^2}. \tag{1.36}$$

(ii)

In the inertial frame $\mathbf{L}$ is time independent so we only need to calculate $\mathbf{L}$ at $t = 0$ we know at $t = 0$,

$$\boldsymbol{\omega}_{\text{inertial}}(0) = \Omega_1^{(0)}\hat{\mathbf{x}} + \Omega_3^{(0)}\hat{\mathbf{z}} \tag{1.37}$$

and as the principle axes of the body line up with $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ the inertia tensor at $t = 0$ in the inertial frame is (in the inertial frame I_{ij} is time dependent as the bodies principle axes rotate),

$$\mathbf{I}_{\text{inertial}}(0) = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \tag{1.38}$$

So the angular momentum in the inertial frame is (this formula is exact)

$$\mathbf{L}_{\text{inertial}} = \mathbf{I}_{\text{inertial}}(0) \cdot \boldsymbol{\omega}_{\text{inertial}}(0) = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \Omega_1^{(0)} \\ 0 \\ \Omega_3^{(0)} \end{pmatrix} = \begin{pmatrix} I_1\Omega_1^{(0)} \\ 0 \\ I_3\Omega_3^{(0)} \end{pmatrix}. \tag{1.39}$$

And the magnitude of the angular momentum is the same as in the rotating frame (1.36).

Marking scheme: The assignment is graded out of 10 with marks for the following:

- One mark for setting ϵ as a quantity proportional to $(I_1 - I_2)$ like in (1.4).
- One mark for obtaining the correct order ϵ^0 equations, something equivalent to equations (1.12-1.14).
- One mark for noting the that solution to the lowest order equation for ω_3 is $\Omega_3 = \Omega_3^{(0)}$.
- One mark for correctly identifying that the solutions to the lowest order equations for ω_2 and ω_1 (equations (1.19) and (1.20)) are sinusoidal oscillations with a frequency $\nu_0 = \frac{I_1 - I_3}{I_3}\Omega_3^{(0)}$.
- One mark for finding the full expressions for $\Omega_1(t)$ and $\Omega_2(t)$ (equations (1.22-1.23).
- One mark for correctly finding the order ϵ equations, (1.25-1.27).
- One mark for finding the solution $\eta_3^{(0)}$ in terms of an integration constant (1.29).
- One mark for correctly implementing the initial conditions on $\eta_3^{(0)}$ to find the constant and the result (1.32).
- One mark for correctly finding $\mathbf{L}$ in the body frame (1.35).
- One mark for correctly finding $\mathbf{L}$ in the inertial frame (1.39)