Marking scheme in red.

1 Question 1:

part a)

(i)

The (gravitational) potential energy is $V(z) = mgz$ so that as usual the Lagrangian for the mass at position $\mathbf{r}(t)$ is

$$\boxed{\mathcal{L} = \frac{1}{2}m_0\dot{\mathbf{r}}^2 - mgz.} \tag{1.1}$$

With the constraint that the mass lies on the rotating hoop described in the question.

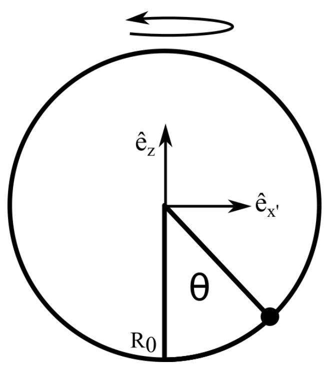
(ii)

Define two unit vectors $\hat{\mathbf{e}}_{x'}, \hat{\mathbf{e}}_{y'}$ that rotate with the plane (the rotation is around the z axis so that z unit vector is same in the rotating frame as in the non rotating frame). In terms of the static unit vectors $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y$ they are ,

$$\hat{\mathbf{e}}_{x'}(t) = \cos(\omega_0 t)\hat{\mathbf{e}}_x + \sin(\omega_0 t)\hat{\mathbf{e}}_y \tag{1.2}$$

$$\hat{\mathbf{e}}_{y'}(t) = -\sin(\omega_0 t)\hat{\mathbf{e}}_x + \cos(\omega_0 t)\hat{\mathbf{e}}_y. \tag{1.3}$$

Note $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{e}}_{x'} = \omega_0\hat{\mathbf{e}}_{y'}$. Inside the rotating plane we can set up the coordinate θ to describe the position of the mass as shown below.



The position vector is then of the form,

$$\mathbf{r}(t) = R_0\left[\sin\theta\hat{\mathbf{e}}_{x'} - \cos\theta\hat{\mathbf{e}}_z\right] \tag{1.4}$$

so that the velocity of the mass is

$$\dot{\mathbf{r}} = R_0\left[\dot{\theta}\left(\cos\theta\hat{\mathbf{e}}_{x'} + \sin\theta\hat{\mathbf{e}}_z\right) + \sin\theta\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{e}}_{x'}\right] \tag{1.5}$$

$$= R_0\left[\dot{\theta}\left(\cos\theta\hat{\mathbf{e}}_{x'} + \sin\theta\hat{\mathbf{e}}_z\right) + \omega_0\sin\theta\hat{\mathbf{e}}_{y'}\right]. \tag{1.6}$$

Therefore the kinetic energy is,

$$T = \frac{1}{2}m_0R_0^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right). \tag{1.7}$$

So that the Lagrangian in the rotating frame is (I have dropped a constant term which does not depend on θ)

$$\boxed{\mathcal{L}' = \frac{1}{2}m_0R_0^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + m_0gR_0\cos\theta.} \tag{1.8}$$

Note there is no Coriolis term above as the Coriolis force is perpendicular to the plane of motion.

part b)

(i)

The equation of motion is

$$\boxed{0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}'}{\partial \dot\theta} - \frac{\partial \mathcal{L}'}{\partial \theta} = m_0 R_0^2 \ddot\theta + m_0 R_0^2 \omega_0^2 \sin\theta \left(\frac{g}{\omega_0^2 R_0} - \cos\theta\right).} \tag{1.9}$$

We have a stationary solution when

$$\ddot\theta = 0 \quad \Rightarrow \quad \sin\theta \left(\frac{g}{\omega_0^2 R_0} - \cos\theta\right) = 0 \tag{1.10}$$

so there are the following stationary solutions

$$\sin\theta = 0 \quad \Rightarrow \quad \theta = 0 \text{ or } \pi \tag{1.11}$$

$$\cos\theta = \frac{g}{\omega_0^2 R_0} \quad \Rightarrow \quad \theta = \pm\,\theta_E \equiv \pm \arccos^{-1}\left(\frac{g}{\omega_0^2 R_0}\right) \text{ so long as } g \le \omega_0^2 R_0^2. \tag{1.12}$$

Now we need to work out which of these stationary solutions support stable oscillations about them. To do this it is convenient to think in terms of an "effective potential" $\tilde V(\theta)$ defined by,

$$\tilde V(\theta) = -m_0 R_0^2 \omega_0^2 \left(\frac{g}{\omega_0^2 R_0}\cos\theta + \frac{1}{2}\sin^2\theta\right). \tag{1.13}$$

The equation of motion in terms of $\tilde V$ is

$$m_0 R_0^2 \ddot\theta = -\frac{\mathrm{d}\tilde V}{\mathrm{d}\theta}. \tag{1.14}$$

The stationary solutions (1.11-1.12) occur at points where $\frac{\mathrm{d}\tilde V}{\mathrm{d}\theta} = 0$ we need to figure out whether these points are minima or maximas of the effective potential. We know when $\omega_0 = 0$ the mass will sit at the bottom of the hoop so $\theta = 0$ is the minimum for small ω_0. When $\omega_0 > \sqrt{g/R_0^2}$ a new there are two new minima which are either side of the center of the hoop at angles $\pm\theta_E$ where the centrifugal force is balanced by gravity. When $\omega_0 > \sqrt{g/R_0^2}$ the position at the bottom of the hoop ($\theta = 0$) has a minima on either side of it so in this case $\theta = 0$ is a maxima. In all cases the point at the top of the hoop $\theta = \pi$ is a maxima of the effective potential. The above can be confirmed mathematically using the second derivative of the effective potential,

$$\frac{\mathrm{d}^2\tilde V}{\mathrm{d}\theta^2} = m_0 R_0^2 \omega_0^2 \left(\sin^2\theta - \cos^2\theta + \frac{g}{\omega_0^2 R_0}\cos\theta\right). \tag{1.15}$$

- When $\theta = 0$ we have

  $$\left.\frac{\mathrm{d}^2\tilde V}{\mathrm{d}\theta^2}\right|_{\theta=0} = m_0 R_0^2 \omega_0^2 \left(-1 + \frac{g}{\omega_0^2 R_0}\right). \tag{1.16}$$

  There are two cases: (I) when $g < \omega_0 R_0^2$ we have $\left.\frac{\mathrm{d}^2\tilde V}{\mathrm{d}\theta^2}\right|_{\theta=0} > 0$ so $\theta = 0$ is a local minimum of $\tilde V$ and (II) when $g > \omega_0 R_0^2$ we have $\left.\frac{\mathrm{d}^2\tilde V}{\mathrm{d}\theta^2}\right|_{\theta=0} < 0$ so $\theta = 0$ is a local maximum.

- When $\cos\theta = \frac{g}{\omega_0^2 R_0}$ we have $\sin\theta = \sqrt{1 - \left(\frac{g}{\omega_0^2 R_0}\right)^2}$ so that

  $$\left.\frac{\mathrm{d}^2\tilde V}{\mathrm{d}\theta^2}\right|_{\theta=\pm\theta_E} = m_0 R_0^2 \omega_0^2 \left(1 - \frac{g^2}{\omega_0^4 R_0^2}\right) \tag{1.17}$$

  so $\left.\frac{\mathrm{d}^2\tilde V}{\mathrm{d}\theta^2}\right|_{\theta=\pm\theta_E} > 0$ when $g < \omega_0 R_0^2$ and we have a minimum of $\tilde V$ otherwise we have a local maximum.

The above analysis implies that the stationary point with lowest effective potential is,

$$\boxed{\theta = \theta_s = \begin{cases} \theta = 0 & \text{for } g > \omega_0^2 R_0 \\ \pm\arccos\left(\frac{g}{\omega_0^2 R_0}\right) & \text{for } g < \omega_0^2 R_0 \end{cases}.} \tag{1.18}$$

(ii)

Write $\theta = \theta_s + \phi$ where θ_s is the stationary solution (1.18) then the equation of motion is

$$m_0 R_0^2 \ddot{\phi} = - \left. \frac{\mathrm{d}^2 \tilde{V}}{\mathrm{d}\theta^2} \right|_{\theta=\theta_s} \phi + \dots \tag{1.19}$$

and so for small oscillations (ϕ small) we have .

$$m_0 R_0^2 \ddot{\phi} \approx - \left. \frac{\mathrm{d}^2 \tilde{V}}{\mathrm{d}\theta^2} \right|_{\theta=\theta_s} \phi \tag{1.20}$$

so putting $\phi(t) = a e^{i\omega t}$ we find

$$\omega = \sqrt{ \frac{1}{m_0 R_0^2} \left. \frac{\mathrm{d}^2 \tilde{V}}{\mathrm{d}\theta^2} \right|_{\theta=\theta_s} } \tag{1.21}$$

substituting in equations (1.16) and (1.17) we have

$$\boxed{\omega = \begin{cases} \pm\omega_0 \sqrt{\frac{g}{\omega_0^2 R_0} - 1} & \text{for } g > \omega_0^2 R_0 \\ \pm\omega_0 \sqrt{1 - \frac{g^2}{\omega_0^4 R_0^2}} & \text{for } g < \omega_0^2 R_0. \end{cases}} \tag{1.22}$$

Question 1 is worth 7 marks: 1 mark for getting a Lagrangian equivalent to that in equation (1.8) (the angle θ can be defined a number of different ways all were accepted as long as they rotated with the plane), 1 mark for an equation of motion equivalent to (1.9), 2 marks for identifying $\theta = 0$ and $\theta = \pm\theta_E$ as in equations (1.11-1.12) or equivalent, 1 mark for identifying that stable equilibrium changes depending on whether or not $g > \omega_0^2 R_)$ and 2 marks for the correct frequencies given in equation (1.22).

2 Question 2:

part a)

The position vector of the mass is

$$\mathbf{r}(t) = r(t) \left[\cos(\omega_0 t) \hat{\mathbf{e}}_x + \sin(\omega_0 t) \hat{\mathbf{e}}_y \right] \equiv r(t) \hat{\mathbf{e}}_r \tag{2.1}$$

therefore

$$\dot{\mathbf{r}} = \dot{r} \hat{\mathbf{e}}_r + r\omega_0 \hat{\mathbf{e}}_\theta \tag{2.2}$$

$$\text{with } \hat{\mathbf{e}}_\theta = -\sin(\omega_0 t) \hat{\mathbf{e}}_x + \cos(\omega_0 t) \hat{\mathbf{e}}_y. \tag{2.3}$$

So the Lagrangian in the rotating frame is

$$\mathcal{L}' = \frac{1}{2} m (\dot{r}^2 + \omega_0^2 r^2) \tag{2.4}$$

and the equation of motion is

$$\boxed{\ddot{r} = \omega_0^2 r} \tag{2.5}$$

part b)

The solution of the equation of motion is

$$r(t) = A_+ e^{\omega_0 t} + A_- e^{-\omega_0 t}. \tag{2.6}$$

So the initial condition $\dot{r}(0) = 0$ is equivalent to

$$\omega_0 (A_+ - A_-) = 0 \quad \Rightarrow \quad A_- = A_+ \tag{2.7}$$

thus $r(t)$ is

$$r(t) = \frac{A_+}{2} \cosh \omega_0 t. \tag{2.8}$$

Written in terms of the initial radius we have

$$\boxed{r(t) = r(0) \cosh \omega_0 t.} \tag{2.9}$$

Question 2 is worth 3 marks: 1 mark for determining the equation of motion (2.5), 1 mark for correctly identifying the general solution (2.6) and 1 mark for imposing the initial conditions to get a relationship like equation (2.7) relating the undetermined parameters.