4th assignment. Physics 306, spring 2017.

We found out many submitted assignments are copious and clear clues of cheating. We learn science and pursue the truth of the universe. We wish students do not do something illogical and not make themselves ashamed. TA's also have a duty to report cheating to the instructor.

1. (a)

$$r(\phi) = r_0/(1 + e\ \cos\phi)$$

$$1 = r_0/(r + e\,r\ \cos\phi)$$

Use $r\cos\phi = x$, $r\sin\phi = y$, 1mark. Very obvious for the next steps.

$$e\,x + \sqrt{x^2 + y^2} = r_0$$

$$x^2 + y^2 = (r_0 - e\,x)^2 = r_0^2 + 2r_0\,e\,x + e^2x^2$$

$$(1 - e^2)x^2 + 2r_0\,e\,x + y^2 = r_0^2 \tag{1}$$

$$(1 - e^2)\left(x + r_0\,e/(1 - e^2)\right)^2 + y^2 = r_0^2 + r_0^2e^2/(1 - e^2) = r_0^2/(1 - e^2)$$

1 mark if students obtain the correct equation before plugging a and b.

Simplify it and obtain the following.

$$\frac{(x + a\,e)^2}{a^2} + \frac{y^2}{b^2} = 1$$

If students proved the opposite way, i.e., from the cartesian coordinate to polar coordinate, you'll also get the full marks. There will be many ways to obtain the the result.

1(b) 1 mark

Let us start from the equation (1). Obviously,

$$2r_0\,x + y^2 = r_0^2,$$

where $e = 1$.

2(a) 2 marks. This problem is straightforward. Will have full marks for the clear proof.

Set $l = r^2\dot\phi$. It is the specific relative angular momentum. The following wiki link might be helpful for students. Notice that l is still constant.

`https://en.wikipedia.org/wiki/Two-body_problem_in_general_relativity`
`https://en.wikipedia.org/wiki/Specific_relative_angular_momentum`

TA's announced it in the tutorials, and it is not too difficult to figure out the mass of an object is not considered in the equation.

Using a chain rule, obtain the followings.

$$\dot r = \frac{d}{dt}(1/u) = -\dot u/u^2 = -\frac{du}{d\phi}\frac{d\phi}{dt}\frac{1}{u^2} = -u'\dot\phi/u^2$$

$$l = r^2\dot\phi$$

Then, the equation (0.3) is

$$u'^2 l^2 + (1 - 2V_0 u)(1 + l^2 u^2) - E^2 = 0$$

Divide by l^2,

$$u'^2 + (1 - 2V_0 u)(1/l^2 + u^2) - E^2/l^2 = 0$$

It is clear we should take the derivative $d/d\phi$ of the above equation, in order to find the solution. Constant term E^2 should be removed, and we must have u'' term in the resulting equation.

$$2u'u'' + (1 - 2V_0 u)2uu' - 2V_0 u'(1/l^2 + u^2) = 0$$

Divide by $2u'$ and obtain

$$u'' + u - \frac{V_0}{l^2} - 3V_0 u^2 = 0 \tag{2}$$

*Instead of setting $l = r^2\dot{\phi}$, this problem can be answered by the following, too. Convert the units back to SI by making the replacements.

$$E \to E/(mc^2)$$
$$l \to l/m$$
$$\dot{r} \to \dot{r}/c$$
$$V_0/r \to V_0/(rc^2)$$
$$V_0 = GM \to GMm$$

2(b)

The equation for the classical solution is

$$u_c'' + u_c - \frac{V_0}{l^2} = 0$$

The classical solution is

$$u_c = \frac{V_0}{l^2} + c_1 \cos\phi + c_2 \sin\phi$$

From the description of the problem, we should write the sum of a sine and a cosine function as the sine function and the phase angle, ϕ_0. In addition, we already know the classical solution must be the ellipse, which we studied in the problem 1 of this assignment.

Therefore, we want to express the classical solution as following.

$$u_c = \frac{V_0}{l^2}(1 + e\ \cos(\phi - \phi_0))$$

1 mark if students find the classical solution in this form. Some students did not obtain the coefficient, $\frac{V_0}{l^2}$. We evaluate -0.5 marks for this mistake.

And from the perturbation, $u = u_c + u_g$, the equation of the motion is

$$u_c'' + u_g'' + u_c + u_g - \frac{V_0}{l^2} - 3V_0(u_c^2 + 2u_c u_g + u_g^2) = 0.$$

It is obtained by plugging $u = u_c + u_g$ into the Eq. (2).

Simplify the equation from the classical equation and perturbation condition, the higher orders of u_g are ignored. $u_g^2 \to 0$

$$u_g'' + u_g - \frac{3V_0^3}{l^4}\,(1 + e\ \cos((\phi - \phi_0)))^2 - \frac{6V_0^2}{l^2}\,(1 + e\ \cos(\phi - \phi_0))u_g = 0$$ 1 mark

There are the other way to solve 2(b). We describe the detail later at (*) below the Eq. (3).

2(c)

The simplified equation by dropping $-3V_0^4/l^4$, and $\cos^2(\phi - \phi_0)$ is

$$u_g'' + u_g - \frac{6V_0^2}{l^2}\,(1 + e\ \cos(\phi - \phi_0))u_g = \frac{6V_0^3}{l^4}\,e\ \cos(\phi - \phi_0))$$

We could drop one more term to simplify and make possible to solve the problem. The 3rd term, $-\frac{6V_0^2}{l^2}\,(1 + e\ \cos(\phi - \phi_0))u_g$, could be ignorable since $\frac{6V_0^2}{l^2 c^4} \ll 1$, and the following resulting equation looks like that of an forced oscillation.

$$u_g'' + u_g = \frac{6V_0^3}{l^4}\,e\ \cos(\phi - \phi_0)) \tag{3}$$ 2 marks

(*) There are different way to reach the Eq. 3. Many students used this approach. When we plug u_c into the Eq. (2), we will obtain $\frac{3V_0^3}{l^4}\,(1 + e\ \cos((\phi - \phi_0)))^2$. By dropping $-3V_0^4/l^4$, and $\cos^2(\phi - \phi_0)$, we make Eq. 3. If students solve them like it, we also give the full marks for 2(b), and 2 marks of 2(c).

We can solve the equation as we have learned in the lecture. I will skip the detail. The particular solution will be obtained from $u_p(\phi) = c_1 + c_2\cos(\phi - \phi_0) + c_3\phi\sin(\phi - \phi_0)$.

As the result, the solution for the perturbation is

$$u_g(\phi) = \frac{3V_0^3(e\cos(\phi - \phi_0) + 2\ e\ \phi\sin(\phi - \phi_0))}{2l^4} + e_2\cos(\phi - \phi_2), \tag{4}$$ 1 mark

where the 2nd term, $e_2\cos(\phi - \phi_2)$, is the homogenous solution. Many students made a mistake to put $\phi_2 = \phi_0$. It is obvious that the homogenous solution should be independent of ϕ_0 and e. We evaluate -0.5 marks for this mistake. It is allowed for Students to express $e_2\cos(\phi - \phi_2)$ as $c_1\cos\phi + c_2\sin\phi$. e_2 is not an eccentricity, and ϕ_2 is not an angle of periastron. We chose them because it makes the solutions look simple.

We assumed u_g be very small, the first term of u_g must be small since $V_0^3/(l^4c^6)$ is small, and e_2 must be small by the definition of the perturbation. Anybody finds this essential restriction? More exactly, e_2 should be much smaller than the prefactor of the classical solution, $\frac{V_0}{l^2}$.

Some stubborn students might wonder why the constant term $-3V_0^4/l^4$ should be dropped if the smaller forcing term $\frac{6V_0^3}{l^4}\ e\ \cos(\phi-\phi_0))$ is not dropped[1] . If we keep the constant term, the equation of the motion is

$$u_g'' + u_g = \frac{3V_0^3}{l^4}(1 + 2\ e\ \cos(\phi-\phi_0))$$

We give students the full 2 marks, too, if students get the equation instead of the Eq. (3).

Actually, it seems not too difficult to solve, compared to the equation without the constant term. Actually, the same type of the particular solution works again. $u_p(\phi) = c_1 + c_2\cos(\phi-\phi_0) + c_3\phi\sin(\phi-\phi_0)$. The solution for this case is

$$u_g(\phi) = \frac{3V_0^3(2 + e\cos(\phi-\phi_0) + 2\ e\ \phi\sin(\phi-\phi_0))}{2l^4} + e_2\cos(\phi-\phi_2), \tag{5}$$

where e_2 is very small.

We give students the full one mark, too, if students get the Eq. (5) instead of the Eq. (4).

Now the full solution is

$$u(\phi) = u_c(\phi) + u_g(\phi),$$

so,

$$u(\phi) = \frac{V_0}{l^2}(1+e\ \cos(\phi-\phi_0)) + \frac{3V_0^3(2 + e\cos(\phi-\phi_0) + 2\ e\ \phi\sin(\phi-\phi_0))}{2l^4} + e_2\cos(\phi-\phi_2). \tag{6}$$

We note that the final solution have two e's and two constant angles ϕ_0, ϕ_2, so we can guess this motion is made of two elliptical rotations. It is called a precession.

Appendix

We are also able to calculate the approximate perihelion advance from the Eq. (6). We find the extremum angle around ϕ_0, $\phi_0 + 2\pi$. Those extrema are $\frac{1}{The\ minimum\ distance}$'s for a rotation and the next rotation. The difference between the extrema is the approximate perihelion advance angle.

Follow the steps.

1. Take the differentiate Eq. (6) with respect to ϕ. Obtain $u'(\phi)$.

2. Taylor expand $u'(\phi)$ around a to ϕ_0.

$$u'(\phi) = u'(\phi_0) + u''(\phi_0)(\phi-\phi_0) + \frac{1}{2}u'''(\phi_0)(\phi-\phi_0)^2 + O((\phi-\phi_0)^3) \tag{7}$$

3. Find the extremum ϕ_{ex} satisfy $u'(\phi_0) + u''(\phi_0)(\phi-\phi_0) + \frac{1}{2}u'''(\phi_0)(\phi-\phi_0)^2 = 0$.

3. Repeat the step 2 and 3 around $\phi_0 + 2\pi$, and then find the extremum ϕ_{ex2}.

4. The approximate perihelion advance angle is $\phi_{ex2} - 2\pi - \phi_{ex}$.

With $e = 0.7$, $e_2 = 0.0001$, $l = 0.5$, $V_0 = 0.01$, $\phi_0 = 0.5$ rad, $\phi_2 = 0.6$ rad, and $\phi = (0, 20\pi)$, we obtain the approximate perihelion advance angle is about 0.00752632 rad.

The graph of the orbit will be as following.

[1]Note that $e < 1$ and $\cos(\phi-\phi_0)) < 1$.

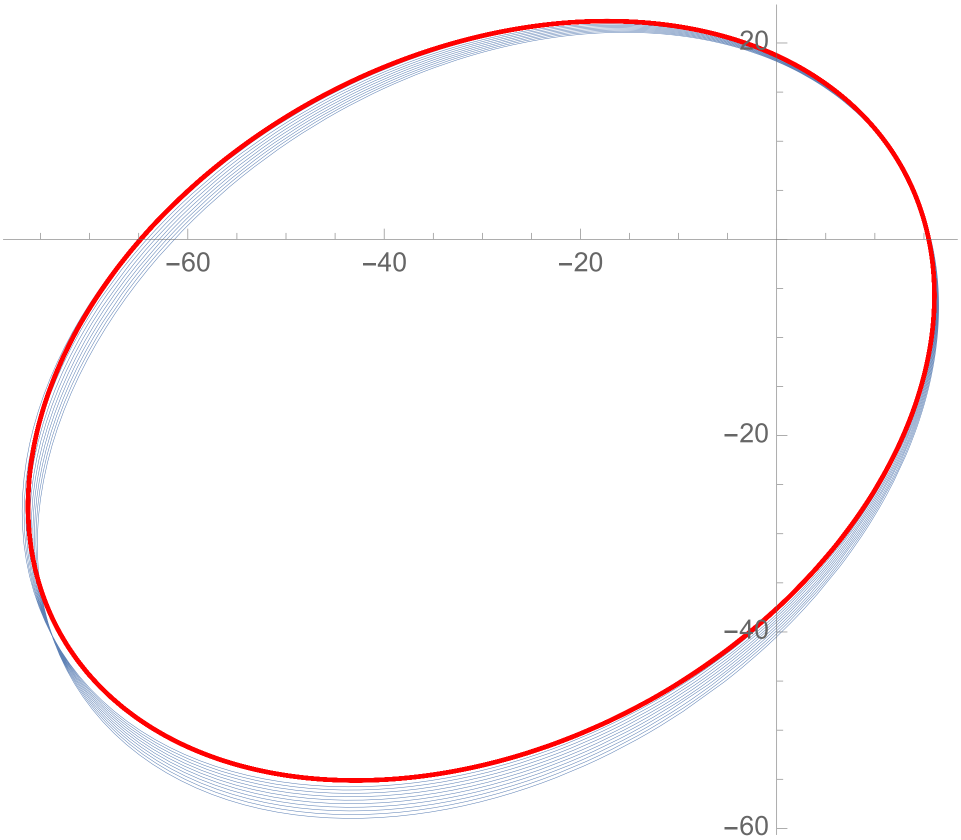

Figure 1: Precession of the perihelion of an object by approximate General Relativity. The red line is the graph of the classical solution. The blue thin graph is the solution for the problem 2 with $e = 0.7$, $e_2 = 0.0001$, $l = 0.5$, $V_0 = 0.01$, $\phi_0 = 0.5$ rad, $\phi_2 = 0.6$ rad, and $\phi = (0, 20\pi)$. Observe precessing during 10 turns. When we increase e_2 gradually, and it becomes same and then bigger than $V_0^2/l^2 = 0.004$, what do we think the graph would look like? There are hints in the problem 1 of this assignment.