Phys 306 Homework 1 solutions. February 7, 2017

Marking scheme in red.

1 Question 1:

part a)

The equation of motion for the position coordinate $q(t)$ is

$$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = f_0\theta(t) \tag{1.1}$$

with

$$f_0 = \frac{F_0}{M} \tag{1.2}$$

$$\omega_0 = \sqrt{\frac{k}{M}} \tag{1.3}$$

$$\gamma = \frac{\eta}{2M}. \tag{1.4}$$

We are interested in the case where $t > 0$ so that $\theta(t) = 1$ so we have to solve

$$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = f_0. \tag{1.5}$$

We know that the solution to (1.5) is of the form,

$$q(t) = p(t) + x(t) \tag{1.6}$$

where $p(t)$ is *any* solution to (1.5) and $x(t)$ is the solution to the equation with out any forcing

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0. \tag{1.7}$$

The solution to the equation (1.7) is given in the lecture notes

$$x(t) = \begin{cases} Ae^{-\gamma t}\cos(\Omega_0 t + \phi) & \text{for } \gamma < \omega_0 \\ e^{-\gamma t}\left(A_+ e^{\Omega_0 t} + A_- e^{-\Omega_0 t}\right) & \text{for } \gamma > \omega_0 \\ (A_1 + A_2 t)e^{-\gamma t} & \text{for } \gamma = \omega_0 \end{cases} \tag{1.8}$$

with $\Omega_0 = \sqrt{|\omega_0^2 - \gamma^2|}$ and $A, A_\pm, A_1, A_2, \phi$ are constants to be determined by the initial conditions. Now we just need to find a single particular solution $p(t)$ to equation (1.5). The easy way to do this is with a physically motivated guess. Note that the complimentary solution $x(t)$ dies down as for long times (in all cases) so that at long times $q(t) = p(t)$ (cf. equation (1.6)). Physically we know that when we apply a constant force to a harmonic oscillator it will change its equilibrium position and that the damping will cause the oscillators position to tend to its equilibrium position. From this we see that $p(t) = c$ (constant) is a good guess for the particular solution. Plugging the guess into (1.5) gives us the equilibrium position

$$c = \frac{f_0}{\omega_0^2}. \tag{1.9}$$

So the solution for $q(t)$ is

$$q(t) = \begin{cases} \frac{f_0}{\omega_0^2} + Ae^{-\gamma t}\cos(\Omega_0 t + \phi) & \text{for } \gamma < \omega_0 \\ \frac{f_0}{\omega_0^2} + e^{-\gamma t}\left(A_+ e^{\Omega_0 t} + A_- e^{-\Omega_0 t}\right) & \text{for } \gamma > \omega_0 \\ \frac{f_0}{\omega_0^2} + (A_1 + A_2 t)e^{-\gamma t} & \text{for } \gamma = \omega_0. \end{cases} \tag{1.10}$$

Now we have to satisfy the initial conditions. The velocity is

$$\dot{q}(t) = \begin{cases} -Ae^{-\gamma t}[\Omega_0 \sin(\Omega_0 t + \phi) + \gamma\cos(\Omega_0 t + \phi)] & \text{for } \gamma < \omega_0 \\ e^{-\gamma t}\left[A_+(\Omega_0 - \gamma)e^{\Omega_0 t} - A_-(\Omega + \gamma)\Omega_0 e^{-\Omega_0 t}\right] & \text{for } \gamma > \omega_0 \\ \left[A_2 - \gamma A_1 + (1-\gamma)A_2 t\right]e^{-\gamma t} & \text{for } \gamma = \omega_0. \end{cases} \tag{1.11}$$

So for zero velocity at $t = 0$ we have:

- In the underdamped case ($\omega_0 > \gamma$):

$$\Omega_0 \sin\phi + \gamma \cos\phi = 0 \quad \Rightarrow \quad \phi = -\tan^{-1}\left(\frac{\gamma}{\Omega_0}\right) \tag{1.12}$$

- In the overdamped case ($\omega_0 < \gamma$):

$$A_+(\Omega_0 - \gamma) - A_-(\Omega_0 + \gamma) = 0 \quad \Rightarrow \quad A_- = -\left(\frac{\gamma - \Omega_0}{\gamma + \omega_0}\right) A_+ \tag{1.13}$$

- In the critically damped case ($\omega_0 = \gamma$):

$$A_2 = \gamma A_1 \tag{1.14}$$

As the oscillator is at rest for $t < 0$ it must start at $q(0) = 0$ which means:

- In the underdamped case ($\omega_0 > \gamma$):

$$\frac{f}{\omega_0^2} + A\cos\phi = 0 \quad \Rightarrow \quad A = -\frac{f_0}{\omega_0 \cos\phi} \tag{1.15}$$

  but from equation (1.12) we know

$$\cos\phi = \frac{\Omega_0}{\sqrt{\Omega_0^2 + \gamma^2}} = \frac{\Omega_0}{\omega_0} \tag{1.16}$$

  therefor

$$A = -\frac{f_0}{\omega_0 \Omega_0} \tag{1.17}$$

- In the overdamped case ($\omega_0 < \gamma$):(using (1.13))

$$0 = \frac{f_0}{\omega_0^2} + \left(\frac{2\Omega_0}{\gamma + \Omega_0}\right) A_+ \tag{1.18}$$

  so that

$$A_\pm = -\frac{(\Omega_0 \pm \gamma) f_0}{2\omega_0^2 \Omega_0}. \tag{1.19}$$

- In the critically damped case ($\omega_0 = \gamma$):

$$A_1 = -\frac{f_0}{\omega_0^2}. \tag{1.20}$$

Putting the everything together we have

$$\boxed{q(t) = \begin{cases} \frac{f_0}{\omega_0^2}\left(1 - e^{-\gamma t}\left[\cos\Omega_0 t + \frac{\gamma}{\Omega_0}\sin\Omega_0 t\right]\right) & \text{for } \omega_0 > \gamma \\ \frac{f_0}{\omega_0^2}\left(1 - e^{-\gamma t}\left[\cosh\Omega_0 t + \frac{\gamma}{\Omega_0}\sinh\Omega_0 t\right]\right) & \text{for } \omega_0 < \gamma \cdot \\ \frac{f_0 t}{\omega_0}\left[1 - e^{-\omega_0 t}(1 + \omega_0 t)\right] & \text{for } \omega_0 = \gamma \end{cases}} \tag{1.21}$$

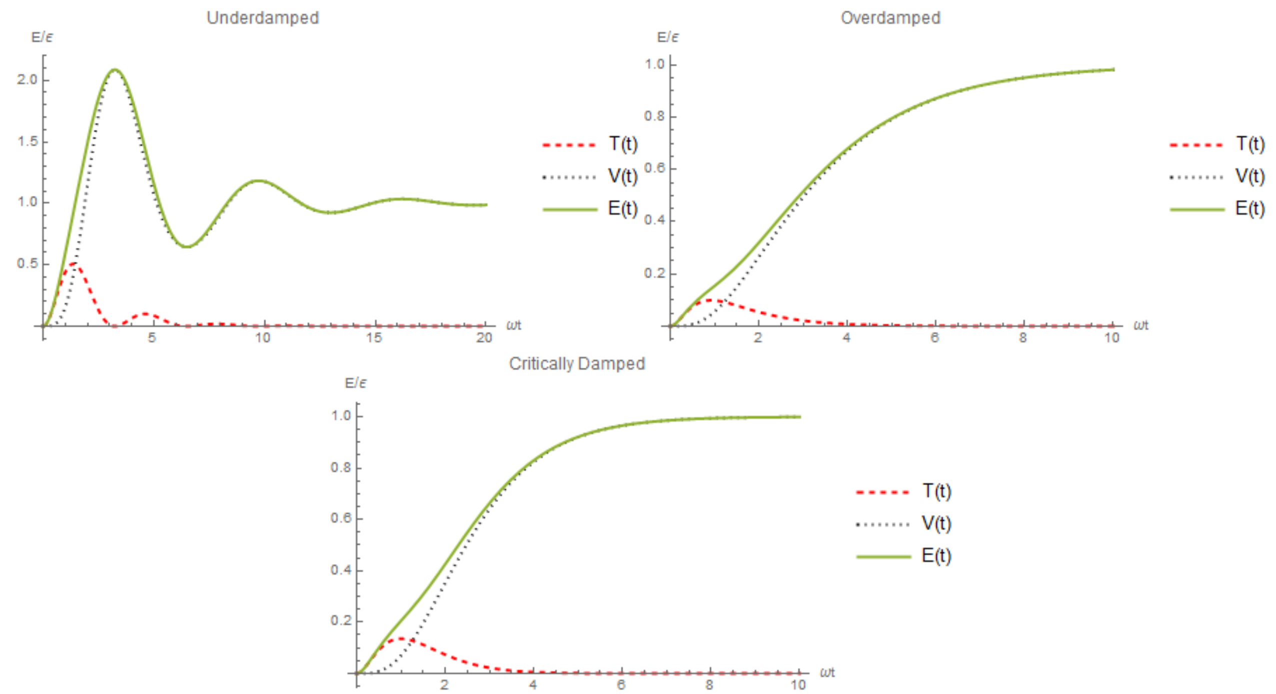
In getting the above simplification I have used $\cos(\Omega_t + \phi) = \cos\phi\cos\Omega_0 t - \sin\phi\sin\Omega_0 t$ to simplify the underdamped solution.

part b)

The kinetic energy,

$$T(t) = \frac{M}{2}\dot{q}^2 = \begin{cases} \frac{Mf_0^2}{2\Omega_0^2}e^{-2\gamma t}\sin^2\Omega_0 t & \text{for } \omega_0 > \gamma \\ \frac{Mf_0^2}{2\Omega_0^2}e^{-2\gamma t}\sinh^2\Omega_0 t & \text{for } \omega_0 < \gamma \\ \frac{1}{2}Mf_0^2 t^2 e^{-2\omega_0 t} & \text{for } \omega_0 = \gamma \end{cases}. \tag{1.22}$$

The potential energy $V = M\omega_0^2[q(t)]^2/2$ with $q(t)$ as in (1.21) (I couldn't find a nice way of simplifying this). The total energy is then $E(t) = T(t) + V(t)$. Plots are shown below. In the plots the energy is scaled by the final energy $\epsilon = \frac{Mf_0^2}{2\omega_0^2}$ which is the total energy added to the system by the external force after a long time. For the underdamped case $\gamma/\omega_0 = 0.25$, for the overdamped case $\gamma/\omega_0 = 1.25$. Note at time $t = 0$ we have $E(0) = V(0) = T(0) = 0$ in all cases and as $t \to \infty$, $T(t) \to 0$ and $E(t) \to V(t) \to \frac{1}{2}k\left(\frac{f_0}{\omega_0}\right)^2$ in all cases.



Question 1 is worth 7 marks: 1 mark for recognising that the solution can be written in the form (1.6), 1 mark for getting that $p(t) = \frac{f_0}{\omega_0^2}$ (or another valid answer), 2 marks for correctly implementing the initial conditions $q(0) = 0 = \dot{q}(0)$, 1 mark for explaining how to go from $q(t)$ to $T(t)$ and $V(t)$, 1 mark for a correct plot and 1 mark for treating all cases (under,over and critically damped).

2 Question 2:

Recall from class the solution for the position of the oscillator in this case is,

$$x(t) = A_0(\omega)\cos(\omega t + \theta(\omega)) \tag{2.1}$$

where $A_0(\omega)$ and $\theta(\omega)$ are given in equations (31) of the course notes (A_0 in this document corresponds to a_0 in the notes and ω here is Ω in the notes). We will only need $A_0(\omega)$ to answer this question,

$$A_0(\omega) = \frac{F_0}{M_0\left((\omega\ {}_3\,\omega_0)^2 + 4\gamma^2\omega^2\right)^{1/2}} \tag{2.2}$$

$\omega_0 = \sqrt{\frac{k}{M_0 s}}$ is the resonance frequency. Suppose the driving frequency is the same as the resonant frequency $\omega = \omega_0$ then A_0 reduces too

$$A_0 = \frac{F}{2M_0\gamma\omega_0}. \tag{2.3}$$

Part i)

The kinetic, potential and total energies are (assuming $\omega_0 = \omega$),

$$T(t) = \frac{M}{2}\dot{x}^2 = \frac{M_0}{2}\omega^2 A^2 \sin^2(\omega t + \theta_0) \tag{2.4}$$

$$V(t) = \frac{k}{2}x^2 = \frac{k}{2}A^2\cos^2(\omega t + \theta_0) = \frac{M_0}{2}\omega_0^2 A^2 \cos^2(\omega t + \theta_0) \tag{2.5}$$

$$E(t) = T(t) + V(t) = \frac{M_0}{2}\omega^2 A^2 \left[\cos^2(\omega t + \theta_0) + \sin^2(\omega t + \theta_0)\right] = \frac{M_0}{2}\omega^2 A^2. \tag{2.6}$$

Part ii)

The rate of energy dissipated per cycle can be calculated by considering the rate at which work is done by the friction force $F_{\text{fr}} = -\eta\dot{x}$,

$$W_{\text{fr}} = -\int_{\text{one cycle}} \mathrm{d}x\eta\dot{x} = -\int_0^{2\pi/\omega} \mathrm{d}t\eta\dot{x}^2. \tag{2.7}$$

The negative sign here just indicates that the friction force is dissipating energy from the system. So the dissipation per cycle is

$$W = \int_0^{2\pi/\omega} \mathrm{d}t\eta\dot{x}^2 = \int_0^{2\pi/\omega} \mathrm{d}t\eta A_0^2\omega^2\sin^2(\omega t + \theta). \tag{2.8}$$

Substitute $s = \omega t$

$$W = 2\pi\eta A_0^2\omega \int_0^{2\pi} \mathrm{d}s \frac{\sin^2(s + \theta)}{2\pi}. \tag{2.9}$$

The integral above is just the average of $\sin^2 s$ over one cycle which is $1/2$ so

$$W = \pi\eta A_0^2\omega = 2\pi\gamma M_0\omega A_0^2 \tag{2.10}$$

Therefore

$$\frac{E}{W} = \frac{\omega_0}{4\pi\gamma} = \frac{Q}{2\pi}. \tag{2.11}$$

Question 2 is worth 3 marks: 1 for something equivalent to equations (2.4-2.6), 1 mark for showing $W = \pi\gamma M_0\omega A_0^2$ and 1 mark for $\frac{E}{W} = \frac{Q}{2\pi}$.
Note that W in part (ii) is not related to the change of total energy as after a long time the total energy is constant because the driving is putting energy into the system at the same rate at which the energy is dissipated.