2nd Assignment

February 2, 2017

1(a)

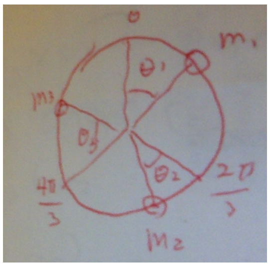

figure 1pt The figure should show detail of how you define your angles, and it should make sense with your initial conditions later when you solve the Lagrange's equations.

We make an ansatz here that the angles are measured from the three equi-spaced points in the figure. Then, positions of 3 masses in polar coordinates, (r, θ)

$$(R_0, \theta_1), (R_0, \theta_2), (R_0, \theta_3)$$

The angels between masses[1] are

$$2\pi/3 + \theta_2 - \theta_1,\ 2\pi/3 + \theta_3 - \theta_2,\ 2\pi/3 + \theta_1 - \theta_3,$$

The tension of springs are acting on the circle. We can solve this system as a periodical, 3 masses, 1 dimensional coupled oscillation.

The potential energy

$$V = \frac{1}{2} k_0 R_0^2 \left((2\pi/3 + \theta_2 - \theta_1)^2 + (2\pi/3 + \theta_3 - \theta_2)^2 + (2\pi/3 + \theta_1 - \theta_3)^2 \right)$$

Minimize potential.

$$\partial_{\theta_1} V = k_0 R_0^2 (2\theta_1 - \theta_2 - \theta_3) = 0$$

$$\partial_{\theta_2} V = k_0 R_0^2 (-\theta_1 - \theta_2 + 2\theta_3) = 0$$

$$\partial_{\theta_3} V = k_0 R_0^2 (-\theta_1 - \theta_2 + 2\theta_3) = 0$$

Therefore, $\theta_1 = \theta_2 = \theta_3 = 2\pi/3$. Our ansatz makes sense. 1 pt for the proper reasoning and result

[1]The relative angles from the 3 spaced points

The kinetic energy is

$$T = \frac{mR_0^2}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right)$$

The Lagrange's equations

$$mR_0^2\ddot{\theta}_1 + k_0R_0^2(2\theta_1 - \theta_2 - \theta_3) = 0$$

$$mR_0^2\ddot{\theta}_2 + k_0R_0^2(-\theta_1 + 2\theta_2 - \theta_3) = 0$$

$$mR_0^2\ddot{\theta}_3 + k_0R_0^2(-\theta_1 - \theta_2 + 2\theta_3) = 0$$

2 (b)

Let $\theta_i = A_i \exp(i\omega t)$

$$-\omega^2 mR_0^2A_1 + k_0R_0^2(2A_1 - A_2 - A_3) = 0$$
$$-\omega^2 mR_0^2A_2 + k_0R_0^2(-A_1 + 2A_2 - A_3) = 0$$
$$-\omega^2 mR_0^2A_3 + k_0R_0^2(-A_1 - A_2 + 2A_3) = 0$$

$$M = \begin{pmatrix} 2k_0R_0^2 - mR_0^2\omega^2 & -k_0R_0^2 & -k_0R_0^2 \\ -k_0R_0^2 & 2k_0R_0^2 - mR_0^2\omega^2 & -k_0R_0^2 \\ -k_0R_0^2 & k_0R_0^2 & 2k_0R_0^2 - mR_0^2\omega^2 \end{pmatrix}$$

$$Det(M) = mR_0^6\omega^2\left(-m^2\omega^4 + 6k_0m\omega^2 - 9k_0^2\right)$$

$$\omega = 0,\ \pm\sqrt{\frac{3k_0}{m}}$$ 1pt

For $\omega = 0$,

$$M = \begin{pmatrix} 2k_0R_0^2 & -k_0R_0^2 & -k_0R_0^2 \\ -k_0R_0^2 & 2k_0R_0^2 & -k_0R_0^2 \\ -k_0R_0^2 & -k_0R_0^2 & 2k_0R_0^2 \end{pmatrix}$$

$\omega = 0$ leads linear motion, so the corresponding solution is

$$(c_1 + c_2t)\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = (c_1 + c_2t)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\omega = \pm\sqrt{\frac{3k_0}{m}}$,

$$M = \begin{pmatrix} -k_0R_0^2 & -k_0R_0^2 & -k_0R_0^2 \\ -k_0R_0^2 & -k_0R_0^2 & -k_0R_0^2 \\ -k_0R_0^2 & -k_0R_0^2 & -k_0R_0^2 \end{pmatrix}$$

Here, we have a condition, $A_1 + A_2 + A_3 = 0$, and two corresponding orthogonal eigenvectors. Without loss of generality, $A_1 = 1$. We can choose two A_2 and A_3 which satisfy $A_2 + A_3 = -1$. From intuition of the initial condition we will meet later and some experience of 1- dimensional 3 masses coupled oscillations, we can choose $A_2 = 0$, and, therefore, $A_3 = -1$. As the result of the choice, the remained normal mode must be $(1, -2, 1)$ because it should be orthogonal to $(1, 1, 1)$ and $(1, 0, -1)$. Because we have $\omega = \pm\sqrt{\frac{3k_0}{m}}$, the resultant oscillation should be the combination of $exp(\sqrt{\frac{3k_0}{m}})$ and $exp(-\sqrt{\frac{3k_0}{m}})$. This can be replaced by the combination of sine and cosine.

Thus, corresponding and orthogonal solutions are

$$(c_3 \cos\sqrt{\frac{3k_0}{m}}t + c_4 \sin\sqrt{\frac{3k_0}{m}}t)\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

and

$$(c_5 \cos\sqrt{\frac{3k_0}{m}}t + c_6 \sin\sqrt{\frac{3k_0}{m}}t)\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

The full solution is (1pt for proper orthogonal normal modes)

$$\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\end{pmatrix} = (c_1+c_2 t)\begin{pmatrix}1\\1\\1\end{pmatrix}+(c_3 \cos\sqrt{\frac{3k_0}{m}}t+c_4 \sin\sqrt{\frac{3k_0}{m}}t)\begin{pmatrix}1\\0\\-1\end{pmatrix}+(c_5 \cos\sqrt{\frac{3k_0}{m}}t+c_6 \sin\sqrt{\frac{3k_0}{m}}t)\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

(i) 1pt for both initial conditions
Stationary condition yields the time derivative initial condition

$$\vec{\dot{\theta}}_0 = \begin{pmatrix}\dot{\theta}_{01}\\\dot{\theta}_{02}\\\dot{\theta}_{03}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

(ii)
From the above figure, we have set three equilibrium points are at $(0, 2\pi/3, 4\pi/3)$. m_1 is at 0, and both m_2 and m_3 are at the opposite position on a circle. I have a positive sign along clockwise direction. By my convention, the angle of m_2 is $+\pi/3$ from the 2nd equilibrium point. Similarly, m_3 placed at $+2\pi/3$ from the 3rd equilibrium point.

$$\vec{\theta}_0 = \begin{pmatrix}\theta_{01}\\\theta_{02}\\\theta_{03}\end{pmatrix} = \begin{pmatrix}0\\\frac{\pi}{3}\\\frac{2\pi}{3}\end{pmatrix}$$

By time derivative initial condition, $c_2 = c_4 = c_6 = 0$, by the angle initial condition, $c_1 = \pi/3$, $c_3 = c_5 = -\pi/6$
The final solution is (1pt)

$$\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\end{pmatrix} = \begin{pmatrix}\pi/3(1-\cos\sqrt{\frac{3k_0}{m}}t)\\\pi/3\\\pi/3(1+\cos\sqrt{\frac{3k_0}{m}}t)\end{pmatrix}$$

If your solution contains cosine and minus cosine with the correct coefficients, you get full marks. For example, if you use $(\pi/3,\ 0, -\pi/3)$ as an initial condition, you will get $\vec{\theta} = \pi/3\cos\sqrt{\frac{3k_0}{m}}t(-1, 0, 1)$. Both solutions physically indicate that one mass is at rest and other two oscillate in opposite direction with same magnitude of the amplitude. These are sum of two normal modes, $(1, 1, 1)$ and $(1, 0, -1)$.

2.
(i) mass : kg, spring constant : kg/s^2 (1pt)
(a) (1pt)

$z_1 : \vec{F} = -k\vec{z}_1 = -2 \times 5 \times 10^3 \; m \cdot kg \;/s^2 \; \hat{z} = -10^4 \; m \cdot kg \;/s^2 \; \hat{z}$
$z_2 : \vec{F} = -k\vec{z}_2 = -8 \times 5 \times 10^3 \; m \cdot kg \;/s^2 \; \hat{z} = -4 \times 10^4 \; m \cdot kg \;/s^2 \; \hat{z}$
(b)
$z_1 : E = \frac{1}{2}kz_1^2 = \frac{1}{2} \times 5 \times 10^3 \times 4m^2 = 10^4 \; J$
$z_2 : E = \frac{1}{2}kz_1^2 = \frac{1}{2} \times 5 \times 10^3 \times 64m^2 = 1.6 \times 10^5 \; J$
$\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{5\times10^3 kg \;/s^2}{2\times10^4 kg}} = 1/2Hz$ (1pt for energy and angular frequency)
$\Delta z(t) = A \cos \omega t$
Kinetic Energy : $K = \frac{1}{2}M\Delta\dot{z}^2 = \frac{1}{2}MA^2\omega^2 \sin^2 \omega t$
Potential energy by gravity is canceled by floating.
Power : $P = \frac{dK}{dt} = \frac{1}{2}M\Delta\dot{z}^2 = \frac{1}{2}MA^2\omega^3 \sin 2\omega t$
Period : $T = 2\pi/\omega$
Mean Power : (1pt for power and mean power)
Because the all kinetic energy are extracted.
$\frac{1}{T}\int_0^T |P| \, dt = \frac{1}{2}MA^2\omega^3 \times \frac{2}{\pi} = 1/2 \times 2 \times 10^4 kg \times 2^2 m^2 \times 1/2^3 Hz^3 \times \frac{2}{\pi} = 3183.1$Watt. [2]

Besides, mean power of sine function is obtained from the root mean square. I will give you full marks for this, too.

$\sqrt{\frac{1}{T}\int_0^T P^2 \, dt} = \frac{1}{2}MA^2\omega^3 \times \sqrt{\frac{1}{2}} = 1/2 \times 2 \times 10^4 kg \times 2^2 m^2 \times 1/2^3 Hz^3 \times \sqrt{\frac{1}{2}} = 3535.53$ Watt. [3]

[2] Mean of $|\sin t|$ is $\sqrt{\frac{2}{\pi}}$. In other words, $\frac{1}{2\pi}\int_0^{2\pi} |\sin t| dt = \sqrt{\frac{2}{\pi}}$

[3] RMS of $\sin t$ is $\sqrt{\frac{1}{2}}$.