Marking scheme in red.

1 Question 1:

part a)

(i)

The Cartesian coordinates of points on a cylinder of radius r are given by,

$$\begin{aligned} x =& r\cos\phi && (1.1)\\ y =& r\sin\phi && (1.2)\\ z =& z && (1.3)\end{aligned}$$

where $0 \le \phi \le 2\pi$ and $-\infty < z < \infty$. So differential changes of all coordinates are related by (r is constant)

$$\begin{aligned} \mathrm{d}x =& \frac{\mathrm{d}x}{\mathrm{d}\phi}\mathrm{d}\phi = -r\sin\phi\,\mathrm{d}\phi && (1.4)\\ \mathrm{d}y =& \frac{\mathrm{d}y}{\mathrm{d}\phi}\mathrm{d}\phi = r\cos\phi\,\mathrm{d}\phi && (1.5)\\ \mathrm{d}z =& \mathrm{d}z. && (1.6)\end{aligned}$$

So that $\mathrm{d}s^2$ becomes,

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = r^2(\cos^2\phi + \sin^2\phi)\mathrm{d}\phi^2 + \mathrm{d}^2z = r^2\mathrm{d}\phi^2 + \mathrm{d}z^2. \tag{1.7}$$

Therefor

$$\boxed{\mathrm{d}s = (r^2\mathrm{d}\phi^2 + \mathrm{d}z^2)^{\frac{1}{2}}}. \tag{1.8}$$

Part (i) is worth 1 mark. Full marks are not awarded unless it is clear how the answer is obtained.

(ii)

The Cartesian coordinates of points on a cylinder of radius r are given by,

$$\begin{aligned} x =& r\cos\phi\sin\theta && (1.9)\\ y =& r\sin\phi\sin\theta && (1.10)\\ z =& r\cos\theta && (1.11)\end{aligned}$$

with $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.Thus we have the following differential relations,

$$\begin{aligned} \mathrm{d}x =& \frac{\partial x}{\partial\phi}\mathrm{d}\phi + \frac{\partial x}{\partial\theta}\mathrm{d}\theta = -r\sin\phi\sin\theta\,\mathrm{d}\phi + r\cos\phi\cos\theta\,\mathrm{d}\theta && (1.12)\\ \mathrm{d}y =& \frac{\partial y}{\partial\phi}\mathrm{d}\phi + \frac{\partial y}{\partial\theta}\mathrm{d}\theta = r\cos\phi\sin\theta\,\mathrm{d}\phi + r\sin\phi\cos\theta\,\mathrm{d}\theta && (1.13)\\ \mathrm{d}z =& \frac{\mathrm{d}z}{\mathrm{d}\theta}\mathrm{d}\theta = -r\sin\theta\,\mathrm{d}\theta && (1.14)\end{aligned}$$

therefor

$$\begin{aligned} \mathrm{d}s^2 =& r^2\Big[\left(\sin^2\phi\sin^2\theta + \cos^2\phi\sin^2\theta\right)\mathrm{d}\phi^2 + \left(\cos^2\phi\cos^2\theta + \sin^2\phi\cos^2\theta + \sin^2\theta\right)\mathrm{d}\theta^2\\ &+ 2\left(\sin\phi\cos\phi\sin\theta\cos\theta - \sin\phi\cos\phi\sin\theta\cos\theta\right)\mathrm{d}\phi\mathrm{d}\theta\Big] && (1.15)\\ =& r^2\left(\sin^2\theta\mathrm{d}\phi^2 + \mathrm{d}\theta^2\right). && (1.16)\end{aligned}$$

So

$$\boxed{\mathrm{d}s = r\left(\sin^2\theta\,\mathrm{d}\phi^2 + \mathrm{d}\theta\right)^{\frac{1}{2}}} \tag{1.17}$$

Part (ii) is worth 1 mark. Full marks are not awarded unless it is clear how the answer is obtained.

(iii)

For Cartesian coordinates

$$\boxed{\mathrm{d}s = (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)^{\frac{1}{2}}}. \tag{1.18}$$

Cylindrical coordinates are defined in equations (1.1-1.3) except now r is allowed to vary so that

$$\mathrm{d}x = \frac{\partial x}{\partial \phi}\mathrm{d}\phi + \frac{\partial x}{\partial r}\mathrm{d}r = -r\sin\phi\,\mathrm{d}\phi + \cos\phi\,\mathrm{d}r \tag{1.19}$$

$$\mathrm{d}y = \frac{\partial y}{\partial \phi}\mathrm{d}\phi + \frac{\partial r}{\partial r}\mathrm{d}r = r\cos\phi\,\mathrm{d}\phi + \sin\phi\,\mathrm{d}r \tag{1.20}$$

$$\mathrm{d}z = \mathrm{d}z \tag{1.21}$$

plugging these into (1.18) and simplifying gives,

$$\boxed{\mathrm{d}s = \left[r^2\mathrm{d}\phi^2 + \mathrm{d}r^2 + \mathrm{d}z^2\right]^{\frac{1}{2}}.} \qquad \text{(cylindrical coordinates)} \tag{1.22}$$

Spherical coordinates are defined in equations (1.9-1.11). We get the following differential relations

$$\mathrm{d}x = \frac{\partial x}{\partial \phi}\mathrm{d}\phi + \frac{\partial x}{\partial \theta}\mathrm{d}\theta + \frac{\partial x}{\partial r}\mathrm{d}r = -r\sin\phi\sin\theta\,\mathrm{d}\phi + r\cos\phi\cos\theta\,\mathrm{d}\theta + \cos\phi\sin\phi\,\mathrm{d}r \tag{1.23}$$

$$\mathrm{d}y = \frac{\partial y}{\partial \phi}\mathrm{d}\phi + \frac{\partial y}{\partial \theta}\mathrm{d}\theta + \frac{\partial y}{\partial r}\mathrm{d}r = r\cos\phi\sin\theta\,\mathrm{d}\phi + r\sin\phi\cos\theta\,\mathrm{d}\theta + \sin\phi\sin\theta\,\mathrm{d}r \tag{1.24}$$

$$\mathrm{d}z = \frac{\partial z}{\partial \theta}\mathrm{d}\theta + \frac{\partial z}{\partial r}\mathrm{d}\,r = -r\sin\theta\,\mathrm{d}\theta + \cos\theta\,\mathrm{d}r \tag{1.25}$$

plugging these into (1.18) and simplifying gives,

$$\boxed{\mathrm{d}s = \left[r^2\left(\sin^2\theta\mathrm{d}\phi^2 + \mathrm{d}\theta^2\right) + \mathrm{d}r^2\right]^{\frac{1}{2}}.} \qquad \text{(Spherical coordinates)} \tag{1.26}$$

Part (iii) is worth 2 marks. One mark for each of the boxed equations. Full marks are not awarded unless it is clear how the answer is obtained.

Part b)

Consider the motion of a particle which starts at (z_1, ϕ_1) at time t_1 and ends at (x_2, ϕ_2) at time t_1, the differentials of the coordinate of the particle may be written

$$\mathrm{d}\phi = \dot{\phi}\,\mathrm{d}t \tag{1.27}$$

$$\mathrm{d}z = \dot{z}\,\mathrm{d}z. \tag{1.28}$$

So that the length of the particles path ℓ is

$$\ell = \int_{(z_1,\phi_1)}^{(z_2,\phi_2)} \mathrm{d}s = \int_{t_1}^{t_2} \mathrm{d}t\sqrt{R_0^2\dot{\phi}^2 + \dot{z}^2} \equiv \int_{t_1}^{t_2} \mathrm{d}t f(\dot{\phi}, \dot{z}, \phi, z). \tag{1.29}$$

We can find the paths for which ℓ is minimised using the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{\phi}}\right) - \frac{\partial f}{\partial \phi} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R_0^2\dot{\phi}}{\sqrt{R_0^2\dot{\phi}^2 + \dot{z}^2}}\right) = 0 \tag{1.30}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{z}}\right) - \frac{\partial f}{\partial z} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{z}}{\sqrt{R_0^2\dot{\phi}^2 + \dot{z}^2}}\right) = 0. \tag{1.31}$$

Integrating both of these with respect to time gives

$$\frac{R_0^2\dot{\phi}}{\sqrt{R_0^2\dot{\phi}^2+\dot{z}^2}} = c_1 \tag{1.32}$$

$$\frac{\dot{z}}{\sqrt{R_0^2\dot{\phi}^2+\dot{z}^2}} = c_2 \tag{1.33}$$

where c_1 and c_2 are constants of integration. Therefore we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = \frac{\mathrm{d}\phi/\mathrm{d}t}{\mathrm{d}z/\mathrm{d}t} = \frac{c_1}{c_2R_0^2} \equiv K. \tag{1.34}$$

Integrating with respect to z,

$$\phi(z) = Kz + b \tag{1.35}$$

K and b are integration constants which we see using the end points,

$$\phi_1 = Kz_1 + b \tag{1.36}$$

$$\phi_2 = Kz_2 + b. \tag{1.37}$$

Subtracting equation (1.36) from (1.37) then rearranging gives

$$K = \frac{\phi_2-\phi_1}{z_2-z_1} \tag{1.38}$$

substituting that back into equation (1.37) then rearranging then gives

$$b = \frac{\phi_1z_2-\phi_2z_1}{z_2-z_1}. \tag{1.39}$$

so that putting everything together we have

$$\boxed{\phi(z) = \frac{(\phi_2-\phi_1)z+\phi_1z_2-\phi_2z_1}{z_2-z_1}.} \tag{1.40}$$

The expression (1.40) solves the Euler-Lagrange equations with the correct end points but it is not the unique solution as we can replace $\phi_2 \to \phi_2 + 2\pi n$ where n is an integer in the expression (1.40) and still get a solution which has the same end points. So the shortest path is

$$\phi(z) = \frac{(\phi_2+2\phi n-\phi_1)z+\phi_1z_2-\phi_2z_1-2\pi nz_1}{z_2-z_1} \tag{1.41}$$

where n is found choseing the value which minimises ℓ,

$$\ell = \int \mathrm{d}s = \int_{\phi_1}^{\phi_2+2\pi n} \mathrm{d}\phi\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}\phi}\right)^2} \tag{1.42}$$

$$= \sqrt{R_0^2+\left(\frac{z_2-z_1}{\phi_2+2\pi n-\phi_1}\right)^2}\int_{\phi_1}^{\phi_2+2\pi n}\mathrm{d}\phi \tag{1.43}$$

$$= \sqrt{R_0^2+\left(\frac{z_2-z_1}{\phi_2+2\pi n-\phi_1}\right)^2}(\phi_2+2\pi n-\phi_1) \tag{1.44}$$

$$= \sqrt{R_0^2(\phi_2+2\pi n-\phi_1)^2+(z_2-z_1)^2}. \tag{1.45}$$

We can see from equation (1.45) that ℓ is minimised when $(\phi_2-\phi_1+2\pi n)^2$ is minimised. When $|\phi_2-\phi_1| = \pi$ are two possible solutions for n (eg if $\phi_2-\phi_1 = \pi$ the minimum n can be either 0 or -1). This reflects the fact that if the two points are on exactly opposite sides of the cylinder then both the clockwise and anticlockwise paths will have the same length.

1b) is worth 4 marks: 1 Mark for deriving a functional for the path length (something like equation (1.29)), 1 mark for deriving the appropriate Eular-Lagrange equation(s) (equations (1.30) and (1.31) above), 0.5 marks for showing that the relation between z and ϕ is linear, 0.5 marks for finding the constants (K and b in the above solution) and 1 mark for correctly discussing the uniqueness of the path.

2 Question 2:

In this case the kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \tag{2.1}$$

and the total potential energy $V_{\rm tot}$ is

$$V_{\rm tot} = W(x_1) + W(x_2) + V(x_1 - x_2). \tag{2.2}$$

So the Lagrangian is

$$L = T - V_{\rm tot} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - W(x_1) - W(x_2) - V(x_1 - x_2). \tag{2.3}$$

and the equations of motion are,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x_1}}\right) - \frac{\partial L}{\partial x_1} = m_1\ddot{x}_1 + \frac{\mathrm{d}W(x_1)}{\mathrm{d}x_1} + \frac{\mathrm{d}V(x_1 - x_2)}{\mathrm{d}(x_1 - x_2)} \tag{2.4}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x_2}}\right) - \frac{\partial L}{\partial x_2} = m_2\ddot{x}_2 + \frac{\mathrm{d}W(x_2)}{\mathrm{d}x_2} - \frac{\mathrm{d}V(x_1 - x_2)}{\mathrm{d}(x_1 - x_2)} \tag{2.5}$$

or

$$\boxed{m_1\ddot{x}_1 = -\frac{\mathrm{d}W(x_1)}{\mathrm{d}x_1} - \frac{\mathrm{d}V(x_1 - x_2)}{\mathrm{d}(x_1 - x_2)}} \tag{2.6}$$

and

$$\boxed{m_2\ddot{x}_2 = -\frac{\mathrm{d}W(x_2)}{\mathrm{d}x_2} + \frac{\mathrm{d}V(x_1 - x_2)}{\mathrm{d}(x_1 - x_2)}}. \tag{2.7}$$

Question 2 is worth 2 marks: 1 for the correct Lagrangian (2.3), 1 mark for the correct equations of motion (2.6) and (2.7).